INTRODUCTIO
IN
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AUCTORE
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PARS ALTERA.

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Typis JOSEPHI ROCCHII
CUM APPROBATIONE.

INTRODUCTIO

IN

ALGEBRAM

PARS ALTERA.

SECTIO III.

CAPUT I.

DE ÆQUATIONUM GENESI, AC PRÆCIPUIS PROPRIETATIBUS.

402. PRopoſitio I. *Ordinare æquationem.*

Reſolutio I. Omnes termini in uno æquationis membro collocentur; poſito o in altero ejuſdem membro.

II. Omnes termini ita collocati illo ordine disponantur, quo exponentes potestatum incognitæ successive decrescunt, videlicet primum locum teneat maxima incognitæ potestas, secundum potestas proxime minor, & ita porro. *Quod erat* &c.

Definitiones.

403. Æquatio tot esse dicitur *dimensionum*, vel *graduum*, quot unitates maximus incognitæ exponens, exhibet. Hinc æquatio *simplex*, quæ uno; *composita*, quæ pluribus constat dimensionum gradibus.

404. Æquatio *completa* dicitur illa, in qua nullus terminus deest, seu in qua incognitæ exponentes per numeros naturales sine interruptione decrescunt usque ad unitatem, deinde ultimus terminus consequitur ex meris quantitatibus cognitis emergens. Ex quo liquet, numerum terminorum in æquatione completa debere unitate excedere exponentem maximæ potestatis incognitæ.

405. Æquatio *incompleta* vocatur illa, in qua unus, vel plures termini completam ingredientes deficiunt; sive in qua series exponentium incognitæ ordine naturali decrescentium in uno, vel pluribus locis interrumpitur.

406. Termini omnes in æquatione ordinata atque completa ex ordine ac loco, quem in ipsa occupant, denominantur. Sic *primus terminus* ille appellatur, in quo incognitæ exponens ad maximam potestatem effertur; *secundus terminus*, in quo incognitæ exponens a primo unitate deficit; *tertius* in

quo

quo dictus exponens comparate ad consimilem exponentem secundi termini unitate decrescit, & sic deinceps usque ad ultimum ex meris quantitatibus cognitis conflatum, qui *homogeneus comparationis* nuncupatur. Eandem quoque subeunt termini deficientes appellationem.

Exemplum. Esto æquatio $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$. Hæc (1.°) est ordinata, termini enim in primo ejus membro sunt ritè dispositi secundum ordinem continuè decrescentem exponentium incognitæ, nimirum 4, 3, 2, 1, & facta est nihilo æqualis. (2.°) Est æquatio quarti gradus, quod maximus incognitæ exponens est 4. (3.°) Est completa, nam in illa exponentium incognitæ successive decrescentium ordo non est interruptus usque ad unitatem, quem terminum excipit ultimus cognitus 24, adeoque quinque terminis, nempe $4+1$, necessario componitur. (4.°) Terminus x^4 dicitur primus, quod incognitæ x exponens 4 est maximus; terminus $10x^3$ vocatur secundus, quod ejusdem incognitæ exponens 3 est unitate minor maximo, & ita porro usque ad terminum absolutè cognitum 24, qui est homogeneus comparationis.

Æquatio quinti gradus $x^5 - x^3 + x^2 - b = 0$ est incompleta, quod termini secundus, & quintus absunt; Analystæ hunc defectum per asteriscos designant, ipsamque sic scribunt, $x^5 * - x^3 + x^2 * - b = 0$.

407. Propositio II. *Æquatio quæcumque tot constat radicibus, quot unitates habet maximus exponens incognitæ.*

 Dem.

Dem. Ejus est æquatio ad zero conditionis, ut quantitas aliqua, sive sit realis quomodocumque expressa, sive sit imaginaria, in locum incognitæ substituta generalem ipsius confirmet characterem, qui exigit, ut omnes termini sese per hujusmodi substitutionem destruant, secus non foret nihilo æqualis, quod repugnat. Ergo æquatio quælibet radicem saltem unam admittit (131.), unumque idcirco factorem, per quem ipsa dividi potest. Hoc in antecessum constituto, esto $x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \ldots$ &c. $= 0$ æquatio generalis, seu indefinita, in qua A, B, C, &c. habentur pro incognitæ x coëfficientibus cum suis signis, quæ æquatio cessat postquam incognitæ exponens evadit $= 0$, velut x^{m-m}, ut patet; idque semel indicasse sufficiat. Hujus æquationis radicem finge, esse $= \alpha$; ergo per ea, quæ vidimus in secunda Sectione, unus ex ejus factoribus erit $x - \alpha$, quare ipsa æquatio exprimetur etiam per $(x - \alpha)(x^{m-1} + A'x^{m-2} + B'x^{m-3} + C'x^{m-4} + \ldots$ &c.$) = 0$, qua divisa per $x - \alpha$, remanet quotus $x^{m-1} + A'x^{m-2} + B'x^{m-3} + C'x^{m-4} + \ldots$ &c. $= 0$. Demonstrabitur igitur eadem plane ratione, istam etiam secundam æquationem unam saltem habere radicem; quæ sit β.

Ejus

Ejus itaque factor erit $x-\beta$, ac proveniet $(x-\beta)$

$(x^{m-2}+A''x^{m-3}+B''x^{m-4}+C''x^{m-5}+\ldots\text{&c.})$ $=0$, atque ita semper, donec incognitæ x maximus exponens ad unitatem perveniat, & ultimus factor designari tunc potest per $x-\pi=0$. Ergo tot factoribus, ac proinde tot radicibus æquatio constabit, ad quot unitates maximus incognitæ ascendit exponens, multiplicatis enim hisce factoribus, quorum unus sit æqualis zero, æquatio data *ex hyp.* reviviscit. *Quod erat* &c.

408. *Coroll.* 1. Ergo $x^m+Ax^{m-1}+Bx^{m-2}$ $+Cx^{m-3}+\ldots\text{&c.}=0$, & $(x-\alpha)(x-\beta)(x-\gamma)$ &c. $=0$ sunt æquationes identicæ, adeoque alterutra sumi potest, vel alteri altera æqualis fieri.

409. *Coroll.* 2. Omnis æquatio composita oriri non modo potest ex æquationibus simplicibus invicem ductis, sed etiam ex aliis æquationibus compositis, quæ tamen & ipsæ simplicibus constare debent; quare omnis æquatio composita vel in æquationes compositas gradus inferioris, vel in simplices resolvi potest. Quævis autem æquatio simplex radicem positivam, vel negativam æquationis compositæ continebit.

410. *Scholion* I. Sit ex. gr. $x=1$, & $x=2$, sive $x-1=0$, $x-2=0$. Si duæ istæ simplices æquationes invicem ducantur, orietur æquatio secundi gradus $x^2-3x+2=0$. Sit præterea $x=3$, sive $x-3=0$, & per hanc æquatio quadratica mul-

tiplicetur; prodibit æquatio tertii gradus $x^3 - 6x^2 + 11x - 6 = 0$; quæ si ulterius in aliam æquationem simplicem $x + 4 = 0$ ducatur, exsurget æquatio quarti gradus $x^4 - 2x^3 - 13x^2 + 38x - 24 = 0$. Ubi animadverte, (1.°) Quamlibet radicem pro incognita x in hac æquatione substitutam ejus terminos destruere; sic faciendo $x = 2$, prodit $16 - 16 - 52 + 76 - 24 = 0$. Quare in quavis æquatione tunc verus aliquis incognitæ valor inventus erit, cum hic in æquationem incognitæ loco introductus eam in nihilum redigit. Profecto si in præcedentibus æquationibus (408.) pro x subrogetur α, fiet $(\alpha - \alpha)(\alpha - \beta)(\alpha - \gamma)$ &c. $= 0$. Quod si γ ponatur quantitas negativa, erit $x + \gamma = 0$; hinc $x = -\gamma$, adeoque substituendo $-\gamma$ pro x; fit rursus productum $(\gamma - \alpha)(\gamma - \beta)(-\gamma + \gamma)$ &c. & cum eo æquatio princeps $= 0$. (2.°) Hanc æquationem esse divisibilem per suos factores $x - 1, x - 2, x - 3, x + 4$, ejusque proinde radices esse $1, 2, 3, -4$, quarum singularum pro incognita x successiva substitutio æquationem reddit nihilo æqualem. (3.°) Æquationem istam biquadraticam esse resolubilem in duas quadraticas $x^2 - 3x + 2 = 0$, $x^2 + x - 12 = 0$, quæ secunda prodit ex multiplicatione æquationum simplicium $x - 3 = 0$, $x + 4 = 0$; itidemque in æquationem cubicam $x^3 - 6x^2 + 11x - 6 = 0$, & in simplicem $x + 4 = 0$, est enim per singulas hasce divisibilis; adeoque (4.°) quamlibet æquationem in tot æquationes ejusdem vel diversi gradus generatim resolvi posse, quot requiruntur ut earum multiplicatio ipsam restituat. (5.°) Radices veras in æquatione

tione cum signo negativo, falsas cum positivo conjungi, quod accidit, quia in æquationibus simplicibus ad zero (quarum productum æquationem compositam gignit) radicum signa mutantur. (6.°) Ex omnibus factoribus æquationem propositi gradus efformantibus unum sufficere, qui ponatur $=0$. (7.°) Eandem incognitam constitutam esse ad omnes radicum valores, quamvis inter se diversos, designandos, quod nihil impedit quominus radices inveniantur; ut vidimus in æquationibus quadraticis.

411. *Scholion* II. Non omittendum, quod si habeatur $ax^n + bx^{n-1} + cx^{n-2} +$ &c. $+ v = 0$, erit etiam $x^n + \frac{b}{a}x^{n-1} + \frac{c}{a}x^{n-2} +$ &c. $+ \frac{v}{a} = 0$; quamobrem statuendo $\frac{b}{a} = A$; $\frac{c}{a} = B$; &c. $\frac{v}{a} = P$; fiet $x^n + Ax^{n-1} + Bx^{n-2} +$ &c. $+ P = 0$. Hujus æquationis radices ponantur p, q, r, &c. ita ut sit $x = p$, $x = q$, $x = r$, &c. stabit $x^n + Ax^{n-1} + Bx^{n-2} +$ &c. $+ P = (x-p)(x-q)(x-r)$ &c. $= (p-x)(q-x)(r-x)$ &c. est enim tam $x-p=0$, &c. quam $p-x=0$, &c. Multiplica per a; fiet $ax^n + aAx^{n-1} + aBx^{n-1} +$&c.$+ aP = a(x-p)(x-q)(x-r)$ &c. seu

ſeu ſuffectis valoribus, $ax^n + bx^{n-1} + cx^{n-2} +$ &c. $+ v = a(x-p)(x-q)(x-r)$ &c.

Si vero fuerit $a + bx + cx^2 + dx^3 = 0$, erit etiam $\frac{a}{d} + \frac{bx}{d} - \frac{cx^2}{d} + x^3 = 0$. Hujus æquationis radices ſint p, q, r; ſtabit [*per dem.*] $\frac{a}{d} + \frac{bx}{d} + \frac{cx^2}{d} + x^3 = d(p-x)(q-x)(r-x)$; ſed $\frac{a}{d} = pqr$ [n. 417.]; ergo multiplicando ſecundum æquationis membrum per $\frac{a}{d} : pqr$, fiet $\frac{a}{d} + \frac{bx}{d} + \frac{cx^2}{d} + x^3 = \frac{ad}{d} \frac{(p-x)(q-x)(r-x)}{p \cdot q \cdot r} = 0 = a(1-\frac{x}{p})(1-\frac{x}{q})(1-\frac{x}{r})$; ſive $a + bx + cx^2 + dx^3 = 0 = a(1-\frac{x}{p})(1-\frac{x}{q})(1-\frac{x}{r})$; & generatim $a + bx + cx^2 + dx^3 + ex^4 +$ &c. $= a(1-\frac{x}{p})(1-\frac{x}{q})(1-\frac{x}{r})(1-\frac{x}{s})\ldots$ &c.

412. *Scholion* III. Non ſemper $a^{\frac{m}{n}} = a^{\frac{mr}{nr}}$, ut alibi (110.) per tranſennā innuimus; etenim $a^{\frac{m}{n}} = \sqrt[n]{a^m}$, adeo-

adeoque quum numerus valorum in omni radicali adæquet numerum expressum per radicis exponentem, talis numerus hic erit $= n$. Sed $a^{\frac{mr}{nr}} = \sqrt[nr]{a^{mr}}$ majorem habet hujusmodi valorum numerum, qui est nr superans n per quantitatem $nr - n$; Ergo si $a^{\frac{mr}{nr}}$ duos habeat valores reales, altera $a^{\frac{m}{n}}$ habere potest omnes valores imaginarios, quod accidit cum a est quantitas negativa, m numerus impar, n, r numeri pares. Exemplo res fiet clarior. Esto $a = -4$, $m = 1, n = 2, r = 2$; erit $a^{\frac{mr}{nr}} = \sqrt[4]{+16} = \sqrt{\pm\sqrt{16}} = \pm\sqrt{\pm 4}$, ubi duo sunt valores reales, ac duo imaginarii; similiter $a^{\frac{m}{n}} = a^{\frac{1}{2}} = \pm\sqrt{-4}$, ubi ambo valores sunt imaginarii. Ratio est quod posita a quantitate negativa, fit semper a^{mr} potestas par, atque a^{m} potestas impar.

413. Propositio III. *Data æquatione in qua primi termini coëfficiens sit unitas, reliquorum coëfficientium legem determinare.*

Resolutio. Esto $x^{n} - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} \ldots \&c. = 0$, cujus coëfficientes A, B, C, &c. cujusque radices cum suis signis sint α, β, γ, &c. Istæ pro x in æquatione successivè substituantur; ex quo sequentes orientur æquationes α^{n}

$$\left.\begin{array}{l}\alpha^n - A\alpha^{n-1} + B\alpha^{n-2} - C\alpha^{n-3} + \&c. \\ \beta^n - A\beta^{n-1} + B\beta^{n-2} - C\beta^{n-3} + \&c. \\ \gamma^n - A\gamma^{n-1} + B\gamma^{n-2} - C\gamma^{n-3} + \&c.\end{array}\right\} = 0$$

Ex earum prima habetur $C\alpha^{n-3} = \alpha^n - A\alpha^{n-1} + B\alpha^{n-2}$ &c. adeoque $C = \alpha^3 - A\alpha^2 + B\alpha$ &c. quare si ex reliquis valores coëfficientis C sic eruantur, & omnes inter se comparentur, fiet

$C = \alpha^3 - A\alpha^2 + B\alpha = \beta^3 - A\beta^2 + B\beta$

$C = \beta^3 - A\beta^2 + B\beta = \gamma^3 - A\gamma^2 + B\gamma$;

Ex his duabus æquationibus elicitur

$B\alpha - B\beta = \beta^3 - A\beta^2 - \alpha^3 + A\alpha^2$

$B\beta - B\gamma = \gamma^3 - A\gamma^2 - \beta^3 + A\beta^2$, ac proinde

$$B = \frac{\beta^3 - \alpha^3 + A\alpha^2 - A\beta^2}{\alpha - \beta} = \frac{\gamma^3 - \beta^3 + A\beta^2 - A\gamma^2}{\beta - \gamma}.$$

Ex hac postrema æquatione per debitas operationes eruitur

$$A = \frac{\alpha\gamma^3 - \alpha\beta^3 - \beta\gamma^3 + \alpha^3\beta + \beta^3\gamma - \alpha^3\gamma}{\alpha^2\beta - \alpha^2\gamma + \beta^2\gamma - \alpha\beta^2 + \alpha\gamma^2 - \beta\gamma^2} \&c. =$$

$\alpha + \beta + \gamma + \&c.$

ut facta per denominatorem multiplicatione verum comperietur. Valorem istum substitue pro coëfficiente A in valore invento coëfficientis B; prodibit

$B =$

$B = \frac{\alpha^2\beta + \alpha^2\gamma - \alpha\beta^2 - \beta^2\gamma}{\alpha - \beta}$ &c. $= \alpha\beta + \alpha\gamma +$ $\beta\gamma +$ &c.

Tandem coëfficientium *A*, & *B* valoribus repertis in valorem coëfficientis *C* introductis, habetur $C = \alpha\beta\gamma$, &c. Ergo in omni æquatione unitate pro primi termini coëfficiente prædita, coëfficiens secundi termini æquatur summæ radicum, mutatis earum signis (410. n. 5.); coëfficiens tertii est æqualis summæ productorum ex binis radicibus, coëfficiens quarti æquatur summæ factorum ex ternis radicibus, mutatis earum signis; & sic deinceps, mutatis semper signis in locis paribus. *Quod erat* &c.

Ut æquationum natura clarius inspiciatur, quinque extruam formulas pro omnibus æquationum cubicarum casibus, quæ signorum tam positivorum quam negativorum combinationes necessarias exhibeant, earum ad gradus superiores continuationem aliis relinquens; sit itaque

Exemplum. Factoribus præmissis invicem ductis, æquationes prodeunt, quæ sequuntur:

$$\left.\begin{array}{l} x-\alpha \\ x-\beta \\ x-\gamma=0 \end{array}\right\} x^3 + x^2(-\alpha-\beta-\gamma) + x(\alpha\beta+\alpha\gamma+\beta\gamma) - \alpha\beta\gamma = 0$$

$$\left.\begin{array}{l} x+\alpha \\ x+\beta \\ x+\gamma=0 \end{array}\right\} x^3 + x^2(\alpha+\beta+\gamma) + x(\alpha\beta+\alpha\gamma+\beta\gamma) + \alpha\beta\gamma = 0$$

$$\left.\begin{array}{l} x-\alpha \\ x-\beta \\ x+\gamma=0 \end{array}\right\} x^3 + x^2(-\alpha-\beta+\gamma) + x(\alpha\beta-\alpha\gamma-\beta\gamma) + \alpha\beta\gamma = 0$$

$x - \alpha$

$$\left.\begin{array}{l} x-\alpha \\ x+\beta \\ x+\gamma=0 \end{array}\right\} x^3+x^2(-\alpha+\beta+\gamma)+x(-\alpha\beta-\alpha\gamma+\beta\gamma)-\alpha\beta\gamma=0$$

$$\left.\begin{array}{l} x-\alpha \\ x+\beta \\ x-\gamma=0 \end{array}\right\} x^3+x^2(-\alpha+\beta-\gamma)+x(-\alpha\beta+\alpha\gamma-\beta\gamma)+\alpha\beta\gamma=0.$$

414. *Coroll.* 1. Cum secundus æquationis terminus deficit, signum est, radices veras esse falsis æquales, sive radicum summam fieri $=0$; secundus enim terminus continet summam radicum sub signo contrario, adeoque etiamsi mutentur signa, summa quantitatum æqualium signo contrario affectarum debet evanescere. Si deficit tertius terminus, productum ex radicibus veris in falsas æquatur producto ex veris in veras, aut falsis in falsas, binis quibuslibet radicibus invicem ductis; & sic de reliquis terminis deficientibus.

415. *Coroll.* 2. Si generatim terminus aliquis evanescat, æquationis radices variis esse debent signis affectæ, secus elisio, adeoque termini evanescentia fieri non posset, ut accidit cum æquationis radices sunt omnes positivæ, aut omnes negativæ.

416. *Coroll.* 3. Si summa radicum verarum summam falsarum excedit, secundus æquationis terminus signum habet negativum; & vicissim si summa radicum falsarum superat verarum summam, secundus terminus signum habet positivum.

417. *Coroll.* 4. Ultimus æquationis terminus est productum ex omnibus radicibus, quæ si fuerint omnes positivæ, earumque numerus sit par, ultimus terminus erit positivus; si dictus numerus sit impar, ul-

ultimus terminus erit negativus; quod radices æquationem sub signo contrario ingrediuntur (410. n. 5.).

418. *Coroll.* 5. Quum itaque, si æquatio sit generatim dimensionis m, omnium radicum positivarum, ac negativarum productum dimensionis m ultimus terminus repræsentet (417.); penultimi termini coëfficiens omnium radicum per numerum $m-1$ combinatarum summam proferet. Antepenultimi coëfficiens omnium radicum per numerum $m-2$ combinatarum summam exhibebit, & ita porro, donec ad secundum terminum omnium radicum summam complectentem perveniatur.

419. *Coroll.* 6. Si inter æquationes simplices æquationem compositam constantes una vel plures adsint nihilo æquales, ut $x-0=0$, vel, quod eodem recidit, si omnes æquationis termini per incognitam multiplicentur, tunc æquatio deprimi potest, idque ad tot gradus, quot sunt hujusmodi factores ipsam multiplicantes. Sic æquatio $x^3+px^2-qx=0$, quamvis tertii gradus appareat, ad secundam pertinet; dividendo enim per x, vel per $x-0$, fit $x^2+px-q=0$; profecto si hæc per x, vel per $x-0$ multiplicetur, redit $x^3+px^2-qx=0$. Eadem ratione in æquatione $x^3-px^2=0$ duo insunt incogniti valores $=0$, & tres in æquatione $x^4-px^3=0$, adeoque prima per duas, altera per tres æquationes deprimenda, & sic de ceteris.

420. *Coroll.* 7. Si radicibus omnibus æqualibus æquatio constet, hæc erit aliqua potestas. Quapropter si æquatio sit dimensionis m, quum radicum æqua-

qualium numerus fit $m\alpha$, secundus terminus erit $\frac{m\alpha}{1}$

per x^{m-1} multiplicatus; tertius terminus summa quadratorum ex omnibus $m\alpha$, quæ binæ sint combinatæ, conflatus proferet numerum $\frac{m \cdot \overline{m-1}}{1 \cdot 2} \alpha^2$ (396.)

per x^{m-2} multiplicatum; quartus terminus includens combinationes quantitatum $m\alpha$, quæ ternæ sint acceptæ, exhibebit numerum $\frac{m \cdot \overline{m-1} \cdot \overline{m-2}}{1 \cdot 2 \cdot 3} \alpha^3$ per

x^{m-3} multiplicatum, & sic deinceps. Hinc si detur binomium $p+q$ ad potestatem indeterminatam m

elevandum, formula generalis fiet $p^m + \frac{m}{1} qp^{m-1} +$

$\frac{m \cdot \overline{m-1}}{1 \cdot 2} q^2 p^{m-2} + \frac{m \cdot \overline{m-1} \cdot \overline{m-2}}{1 \cdot 2 \cdot 3} q^3 p^{m-3} +$ &c.

Quod erat, directe demonstrandum. Sed alibi etiam generalius.

421. *Coroll.* 8. Si æquatio plures habeat radices æquales, quarum numerus m, cum inæqualibus mixtas, ultimus terminus multiplum hujusmodi radicum æqualium m exhibebit; penultimi termini coëfficiens multiplum radicum æqualium, quarum numerus $m-1$,

con-

continebit; antepenultimi coëfficiens multiplum talium radicum, quarum numerus $m-2$ complectetur; & ita porro, donec ad unam tantum radicem ex æqualibus perveniatur, seu donec sit $m-n=1$. Sic si septem dimensionum æquatio habeat radices a, a, a, a, b, c, d; quum ultimus terminus esse debeat a^4bcd, erit $m=4$, ac proinde $a^4 \times bcd$ erit multiplum potestatis a^4; penultimi autem coëfficiens constat septem hisce radicibus per numerum senarium combinatis, ubi saltem a^3 reperiri debet, quare hujusmodi coëfficiens multiplum potestatis a^3,

seu a^{m-1} includet. In termini antepenultimi coëfficiente quum omnium radicum, quæ quinæ sint adsumptæ, summa lateat, patet, in hujusmodi combinationibus adesse necessario debere quadratum aa, adeoque termini antepenultimi coëfficiens multiplum

quadrati a^2, seu a^{m-2} comprehendent, & sic deinceps.

Hinc si plures æquatio habeat radices æquales diversi generis cum inæqualibus commixtas, ex. gr. $a, a, a, a, a, b, b, b, b, c, c, c, d, e, f$, eadem regula locum habet, & idcirco penultimi termini

coëfficiens multiplum constituet quantitatum a^{5-1},

b^{4-1}, c^{3-1}, & sic deinceps; ubi perspicitur, coëfficientem quarti ab ultimo termini comprehendere

multiplum quantitatum a^{5-3}, b^{4-3}, nullum vero

 quan-

quantitatis c, quia radix c ingredi nequit omnes partes ipsum quartum terminum conflantes, quod

etiam statim dignoscitur ob $c^{3-3}=1$.

422. *Coroll.* 9. Æquatio rationalis, seu quæ omnibus constat terminis rationalibus, habere non potest factorem $A \pm B\sqrt{-1}=0$, nisi habeat alterum $A \mp B\sqrt{-1}$, secus non esset rationalis contra hypothesim, nullus enim alius factor præter secundum ductum in primum facere potest productum rationale. Mox idem generaliter, ac directe demonstrabitur.

423. *Scholion.* Interea mihi liceat hanc quoque demonstrationem exhibere. Quoniam expressio quæcumque imaginaria designatur per $x+m+n\sqrt{-1}$ (82.), alia expressio imaginaria, ex cujus multiplicatione ipsa rationalis evadere debeat, sit $x+p+q\sqrt{-1}$, in qua indeterminatæ p, q sunt determinandæ. Fiat hujusmodi expressionum productum, quod erit

$$x^2+mx+nx\sqrt{-1}+px+pm-qn+pn\sqrt{-1}$$
$$+\ qx\sqrt{-1} \qquad\qquad\qquad +qm\sqrt{-1}.$$

Hoc vero esse nequit rationale, nisi quantitates imaginariæ evanescant; quamobrem ut p cum m, & q cum n comparari possint, fiat $nx\sqrt{-1}+qx\sqrt{-1}=0$, & $pn\sqrt{-1}+qm\sqrt{-1}=0$; ex quo $q=-n$, & $p=-\frac{qm}{n}=m$; hinc $x+p+q\sqrt{-1}$ evadit $x+m-n\sqrt{-1}$.

424.

424. *Coroll.* 10. Omnis æquatio rationalis, & ordinis imparis unam saltem habet radicem realem, & si plures habeat radices reales, illæ erunt semper numero impares; si enim haberet omnes factores formæ $A + B\sqrt{-1}$, vel $A - B\sqrt{-1}$, ultimus terminus foret necessario irrationalis contra hypothesim.

425. *Coroll.* 11. Si æquatio ordinis paris habeat omnes radices imaginarias, vel omnes radices æquales tam ejusdem, quam diversi generis numero in quovis genere pares; habebit ultimum terminum positivum; sumptis enim duobus factoribus $A + B\sqrt{-1}$, $A - B\sqrt{-1}$, eorum productum est $A^2 + B^2$, & omnes potestates pares sunt semper affirmativæ.

426. *Coroll.* 12. Hinc quum æquatio ordinis paris, quæ habeat ultimum terminum negativum, habere nequeat omnes radices imaginarias, duas saltem habebit radices reales, alteram positivam, alteram negativam.

427. *Coroll.* 13. Quod si æquatio ordinis imparis habeat ultimum terminum negativum, unam saltem radicem positivam continebit.

428. *Corollarium* 14. Quoniam radix imaginaria $A + B\sqrt{-1}$ per aliam $A - B\sqrt{-1}$ multiplicata dat productum affirmativum $A^2 + B^2$, etiamsi A, B sint quantitates negativæ, idque semper evenit, si quotlibet hujusmodi paria invicem ducentur; sequitur, æquationem radices omnes imaginarias complectentem semper habere terminorum omnium aggregatum af-

affirmativum; quapropter si in istis æquationibus substituatur pro incognita quantitas quævis rationalis affirmativa, vel negativa, terminorum aggregatum semper erit affirmativum. Multiplicentur enim inter se

duo factores $x - A - B\sqrt{-1}$, $x - A + B\sqrt{-1} = 0$, fit $x^2 \pm 2Ax + A^2 + B^2 = 0$, in qua æquatione si pro x substituatur $+c$, vel $-c$, patet terminorum aggregatum (ob quadratum $c^2 \pm 2Ac + A^2$) semper fore affirmativum, & ita semper, quicumque sit talium factorum numerus, nam quantitas $c^2 \pm 2Ac + A^2 + B^2$ ducta in quamlibet quantitatem affirmativam semper gignit productum affirmativum.

429. *Coroll.* 15. Igitur si substitutis in æquatione pro incognita pluribus valoribus singula terminorum aggregata evadant partim affirmativa, partim negativa, ibi radices reales necessario adesse, concludendum est; nimirum radix realis inter duos valores consistit, quorum in æquationem pro incognita introductio duo terminorum aggregata diversis signis reddit affecta; ac proinde isti duo valores sunt ipsius radicis *limites*, qui quo plus ad radicis valorem accedent, eo magis cum ipsa congruent, donec si arctissimi evadant, pro ipsa sine errore sensu percipiendo haberi possunt, quod accidit, cum nullæ radices rationales inveniuntur, quamvis certum sit, ob dictam signorum diversitatem, radicem realem in æquatione contineri, adeoque radicem istam esse irrationalem. Id etiam directe alibi demonstrabitur.

430. *Coroll.* 16. Quod si valoribus quibuscumque pro incognita in æquatione suffectis, singula termino-

norum aggregata fiant semper affirmativa, neque aliter evenire possit, certum erit, æquationem, vel ex radicibus omnibus imaginariis, vel ex pari radicum æqualium ejusdem aut diversi generis numero tantummodo conflatam esse.

431. *Coroll.* 17. Denique si pluribus valoribus pro incognita in æquatione substitutis, tot in dictis terminorum aggregatis prodeant signorum permutationes, quot habet æquatio dimensiones, ista radices omnes reales complectetur. Ut plurimum quot erunt in aggregatis hisce permutationes, tot aderunt in æquatione proposita radices reales; reliquæ vero ad æquationis gradum complendum necessariæ vel erunt imaginariæ, vel invicem æquales, quarum tamen numerus par erit. Dixi *ut plurimum*; potest enim accidere, ut inter duos numeros in æquatione pro incognita substitutos duæ vel plures radices reales delitescant, quamvis nulla ex hisce substitutionibus occurrat signorum mutatio. Quomodo autem hujusmodi radicum limites inveniantur, suo loco docebimus.

432. *Coroll.* 18. Hinc rursus (1.°) æquatio quævis $x^m + px^{m-1} + qx^{m-2} + \&c. \ldots - P = 0$, cujus ultimus terminus est negativus, una saltem radice reali positiva gaudebit. Fac enim $x = 0$, remanebit $-P$; pone $x = \infty$, prodibit $+x^m$, ceteri siquidem termini comparatè ad ∞^m evanescunt. Limites igitur erunt 0, & $+\infty$; sed termini omnes inter 0,

& $+\infty$ intercedentes sunt positivi (18. n. 2.); ergo &c.

(2.°) Æquatio proposita gradus disparis, cujus ultimus terminus sit positivus, unam saltem radicem, quam habet realem (424.), habebit quoque negativam. Etenim mutatis signis in locis paribus, ultimus terminus evadet negativus, adeoque per num. præced. una saltem radice positiva constabit æquatio, quæ radix idcirco in proposita est negativa, ut ex Prop. I. Cap. subsequentis clarius intelligetur.

(3.°) Æquatio gradus paris; cujus ultimus terminus est negativus, duas saltem radices reales, alteram positivam, negativam alteram possidet. Primum patet ex num. (1.°). Secundum agendo ut in num. (2.°) fit manifestum; quæ cum supra demonstratis (425. 426. 427.) congruunt.

433. Propositio IV. *Summas omnium potestatum, ad quas æquationis cujuscumque radices reales efferri possunt, invenire.*

Resolutio. Esto $o = x^m - px^{m-1} + qx^{m-2} - rx^{m-3}$ + &c. cujus factores $x - a = o$, $x - b$, $x - c$, &c. seu $1 - \frac{a}{x} = o$, $1 - \frac{b}{x}$, $1 - \frac{c}{x}$, &c. adeoque a, b, c, &c. æquationis radices. Ponatur $x = \frac{1}{y}$; æquatio evadet $o = \frac{1}{y^m} - \frac{p}{y^{m-1}} + \frac{q}{y^{m-2}} - \frac{r}{y^{m-3}}$ + &c.; quare multiplicando per y^m, habetur

$o = 1$

$0 = 1 - py + qy^2 - ry^3 + \&c. = (1-ay)(1-by)(1-cy)\ \&c.$ sumptisque logarithmis, erit $l(1-py+qy^2-ry^3+\&c.) = l(1-ay)+l(1-by)+l(1-cy)+\ \&c.$ quibus in seriem resolutis (quoniam $l(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \&c.$ (258.), ob $-z = -py + qy^2 - ry^3 + \&c. = y(-p+qy-ry^2+\&c.)$; $-\frac{z^2}{2} = \frac{y^2}{2}(-p+qy-ry^2+\&c.)^2$; $-\frac{z^3}{3} = \frac{y^3}{3}(-p+qy^2-ry^3+\&c.)^3$; nec non ob $l(1-ay) = -ay - \frac{a^2y^2}{2} - \frac{a^3y^3}{3} - \&c.\ \&c.$) nova fiet æquatio

$$y(-p+qy-ry^2+\&c.)+\frac{y^2}{2}(-p+qy-ry^2+\&c.)^2 + \frac{y^3}{3}(-p+qy-ry^2+\&c.)^3\ \&c. = -y(a+b+c+\&c.) - \frac{y^2}{2}(a^2+b^2+c^2+\&c.) - \frac{y^3}{3}(a^3+b^3+c^3+\&c.)\ \&c.$$

sive mutatis signis

$$\left.\begin{matrix} py - qy^2 + ry^3\ \&c. \\ +\frac{p^2y^2}{2} + \frac{p^3y^3}{3}\ \&c. \\ -\frac{2pqy^3}{2}\ \&c. \\ \&c. \end{matrix}\right\} = \left\{\begin{matrix} (a+b+c+\&c.)y = Ay \\ +\frac{1}{2}(a^2+b^2+c^2+\&c.)y^2 = \frac{1}{2}By^2 \\ +\frac{1}{3}(a^3+b^3+c^3+\&c.)y^3 = \frac{1}{3}Cy^3 \\ \&c. \end{matrix}\right.$$

Terminis igitur homologis inter se conlatis, invenitur

$Ay = py$, seu $A = p$; $\frac{1}{2} By^2 = \frac{p^2 y^2}{2} - qy^2$, seu

$B = p^2 - 2q$; $\frac{1}{3} Cy^3 = \frac{p^3 y^3}{3} - \frac{2pqy^3}{2} + ry^3$, seu

$C = p^3 - 3pq + 3r$; &c. Habemus ergo valores quæsitos tam quoad radicum aggregatum, quam quoad aggregatum quadratorum, cuborum, &c. earundem radicum; *Quod erat* &c.

434. *Coroll.* Ut isti valores ad formulas, quas Newtonus exhibuit in sua Arithmetica universali, reducantur, quoniam $A = p$, erit $B = Ap - 2q$; $p^3 - 3pq = p(p^2 - 2q) - pq = p(Ap - 2q) - Aq$; quapropter $C = p^3 - 3pq + 3r = Bp - Aq + 3r$; &c. nimirum si radicum sub signo contrario summa ponatur $= A$, habebimus

$A = p$
$B = Ap - 2q$
$C = Bp - Aq + 3r$
$D = Cp - Bq + Ar - 4s$
$E = Dp - Cq + Br - As + 5t$
&c.

Cujus progressionis lex statim in conspectum venit.

435. Propositio V. *Si æquatio terminis omnibus fractione qualibet carentibus constet, nullas habere potest radices fractas.*

Dem. Esto generatim $x^r + ax^{r-1} + bx^{r-2} + cx$

$cn^{r-3} + \&c. = 0$, ubi, si fieri potest, sit $n = \frac{m}{n}$, & m, n sint numeri inter se primi, seu numeri, qui nullum habeant communem divisorem præter unitatem. Suffecto pro x valore $\frac{m}{n}$, æquatio evadet $\frac{m^r}{n^r} + \frac{am^{r-1}}{n^{r-1}} + \frac{bm^{r-2}}{n^{r-2}} + \frac{cm^{r-3}}{n^{r-3}} + \&c. = 0$; factaque per n^{r-1} multiplicatione, oritur $\frac{m^r}{n} + am^{r-1} + bm^{r-2}n + cm^{r-3}n^2 + \&c. = 0$; quare $\frac{m^r}{n} = - am^{r-1} - bm^{r-2}n - cm^{r-3}n^2 - \&c.$ Sed n esse nequit [*ex hyp.*] divisor exactus potestatis m^r, quia dividere non potest m; ergo $\frac{m^r}{n}$ est semper fractio. At alterum æquationis membrum $- am^{r-1} - bnm^{r-2} - cn^2m^{r-3} - \&c.$ est numerus integer; ergo prodiret fractio æqualis numero integro, quod est absurdum; *Quodque erat* &c.

436.

436. Propositio VI. *Si* $a+b\sqrt[n]{A}=c+d\sqrt[n]{B}$; *dico fore* $a=c$, & $b\sqrt[n]{A}=d\sqrt[n]{B}$.

Dem. Quoniam $a+b\sqrt[n]{A}=c+d\sqrt[n]{B}$, erit etiam $a-c=d\sqrt[n]{B}-b\sqrt[n]{A}$, quod est absurdum, differentia enim duarum vel plurium quantitatum irrationalium esse non potest æqualis quantitati rationali; quare ut servetur æquatio proposita, fieri debet $a=c$, & $b\sqrt[n]{A}=d\sqrt[n]{B}$; *Quod erat* &c.

437. *Coroll.* 1. Ergo si quodvis polynomium terminis constans rationalibus & irrationalibus sit $=0$; tam rationalium, quam irrationalium summa erit seorsum $=0$; quod etiam valet de terminis realibus cum imaginariis commixtis in pari casu.

438. *Coroll.* 2. Si fuerit $a+b\sqrt{-1}=1$, quantitas imaginaria $b\sqrt{-1}$ evanescet; fac enim ut sit $a+b\sqrt{-1}=1+0\sqrt{-1}$, erit $a=1$, $b\sqrt{-1}=0\sqrt{-1}=0$.

439. Propositio VII. *Qualibet æquatio* $0=\pm T-Nx+Mx^2-Lx^3+$ &c. *divisorem habens* $x+m-n\sqrt{p}=0$, *habet etiam divisorem* $x+m+n\sqrt{p}=0$.

Dem. Quoniam $x=-m+n\sqrt{p}$; substitutis in proposita valoribus, proveniet $\pm T-N(-m+n\sqrt{p})+M(-m+n\sqrt{p})^2-L(-m+n\sqrt{p})^3+$ &c. $=0$, sive potestates evolvendo, terminosque rationales ab irrationalibus secernendo, fient (437.) duæ æquationes

Nm

$Nm + M(m^2 + n^2p) + L(m^3 + 3mn^2p) +$ &c. $\pm T = o$;
$-Nn\sqrt{p} - M(2mn\sqrt{p}) + L(-3m^2n\sqrt{p} - n^3p\sqrt{p}) +$ &c. $= o$;
quæ secunda semper remanet $= o$, etiamsi mutentur signa; quare primam & secundam cum signis mutatis summando, vel, quod eodem recidit, secundam a prima subtrahendo, habetur in residuo
$N(m + n\sqrt{p}) + M(m + n\sqrt{p})^2 + L(m + n\sqrt{p})^3 +$ &c. $= \mp T$ æquatio, quæ propositæ æquivalebit. Sed ista, ac proinde terminus etiam cognitus T, est per $m + n\sqrt{p}$ divisibilis; ergo etiam proposita erit per $m + n\sqrt{p}$ divisibilis. *Quod erat* &c.

440. *Coroll.* Idem demonstratur, si proposita divisorem habeat $x + m - n\sqrt{-p}$; quamobrem esse nequit $m - n\sqrt{\pm p}$ radix æquationis cujuslibet rationalis, nisi adsit alia radix $m + n\sqrt{\pm p}$, & vicissim. Similiter si æquationem ingrediatur radix $-m - n\sqrt{\pm p}$, ipsam ingredi debet alia radix $-m + n\sqrt{\pm p}$, & vicissim; adeoque rursus si æquatio quævis radices imaginarias contineat, erunt istæ semper numero pares.

441. Propositio VIII. *Si æquatio omnes habeat radices reales, ac falsas omnia terminorum signa erunt similia; si reales ac veras, per vices signa dissimilia procedent;* & vicissim.

Dem. Primum patet, æquatio enim exurgit ex factoribus $x + a = o, x + b, x + c$, &c. Secundum sic demonstro. Factores $x - a = o, x - b$ invicem ducti exhibent $x^2 - ax + ab = o$. Æquatio ista multiplicetur per $x - c$,
$$\begin{array}{l} x^2 - ax + ab \\ \quad - bx \end{array}$$
prodibit
$$\begin{array}{l} x^3 - ax^2 + abx - abc = o \\ \quad\;\; - bx^2 + acx \\ \quad\;\; - cx^2 + bcx \end{array}$$
, ubi signa alterna conspiciuntur, & ita semper; *Quod erat* &c.

442.

442. *Scholion.* Generatim quot adsunt signorum consecutiones, tot habet æquatio radices falsas; quot autem occurrunt permutationes, tot continet radices veras, dummodo nullas æquatio comprehendit radices imaginarias. Id vero alibi, ordine sic exigente, demonstrabitur.

CAPUT II.

DE ÆQUATIONUM TRANSFORMATIONE.

443. PRopositio I. *Æquationis radices veras in falsas, & falsas in veras, dummodo imaginariæ non adsint, commutare.*

Resolutio. In locum incognitæ alia vel eadem incognita cum signo negativo sufficiatur; quum enim omnes æquationis radices incognita repræsentet, radicum illarum signa mutentur, necesse est; seu quod eodem recidit, negativæ fient positivæ, & vicissim. *Quod erat* &c.

444. *Corollarium.* Ergo sufficit signa mutare in æquationis propositæ locis paribus. Etenim si æquatio est gradus imparis, mutando post substitutionem signa, loca imparia redeunt positiva ut in proposita, & paria remanent ob exponentes pares immutata. Si æquatio est gradus paris incognita post substitutionem in locis imparibus ob exponentes pares idem signum ut in proposita retinet, in paribus minime. Sit ex. gr. $x^3 - x^2 - 14x + 24 = 0$. Pro x substitue $-y$; emerget $-y^3 - y^2 + 14y + 24 = 0$, sive $y^3 + y^2 - 14y - 24 = 0$; vel propositæ incognitam retinendo,

$x^3 +$

$x^3 + x^2 - 14x - 24 = 0$ ubi ſigna in locis paribus reperiuntur mutata. Quod ſi proponatur $x^4 - 2x^3 - 13x^2 + 38x - 24 = 0$, cujus radices ſunt 1, 2, 3, -4; ſumendo negativè incognitam x, habetur $x^4 + 2x^3 - 13x^2 - 38x - 24 = 0$, cujus radices $-1, -2, -3, +4$.

Nec refert, an terminus aliquis in æquatione deficiat, idem namque terminus ſemper deerit, etiamſi radicum ſigna mutentur (414.). Sic in æquatione $x^3 * - 19x + 30 = 0$, cujus radices ſunt 2, 3, -5, ſi ob ſecundi termini abſentiam mutetur tantummodo ſignum in quarto termino, fiet $x^3 * - 19x - 30 = 0$, cujus radices ſunt $-2, -3, +5$ (410.n.4.).

445. Propoſitio II. *Radices æquationis augere, aut minuere quantitate data.*

Reſolutio. Sit æquatio generalis $x^m + px^{m-1} + qx^{m-2} + rx^{m-3} + \&c. = 0$. Fiat $x - a = y$, ſeu $x = y + a$, atque a ponatur quantitas poſitiva, aut negativa. Peractis ſubſtitutionibus, proveniet

$$x^m = y^m + may^{m-1} + \frac{m.\overline{m-1}}{1.\ 2}a^2y^{m-2} + \frac{m.\overline{m-1}.\overline{m-2}}{1.\ 2.\ 3}a^3y^{m-3} + \&c.$$

$$px^{m-1} = +py^{m-1} + \overline{m-1}\cdot pay^{m-2} + \frac{\overline{m-1}.\overline{m-2}}{1.\ 2}pa^2y^{m-3} + \&c.$$

$$qx^{m-2} = +qy^{m-2} + \overline{m-2}\,qay^{m-3} + \&c.$$

$$rx^{m-3} = +ry^{m-3} + \&c.$$

&c.

Quare ſi exponenti m valor aliquis adſcribatur, ſequentia reperientur

I.

I. Quantitates ultimum æquationis transformatæ terminum componentes cum æquatione proposita prorsus conveniunt, hoc tantum cum discrimine, quod a stet pro x, signa vero varient pro valore ipsius a positivo aut negativo. Si enim fiat $m = 3$, erit

$$\frac{m.\overline{m-1}.\overline{m-2}}{1.\quad 2.\quad 3} a^3 y^{m-3} = a^3;\ \frac{\overline{m-1}.\overline{m-2}}{1.\quad 2} pa^2 y^{m-3}$$

$= pa^2$; $\overline{m-2}\, qay^{m-3} = qa$; $ry^{m-3} = r$; nempe

$a^3 + pa^2 + qa + r = o$, seu $a^3 + pa^{3-1} + qa^{3-2} +$

$ra^{3-3} = o$, quæ cum proposita æquatione generali, posita $m = 3$, perfecte congruit.

II. Si ultimi æquationis transformatæ termini jam inventi portiones singulæ multiplicentur per exponentes speciei a, & facta per eandem a dividantur, *coëfficiens* obtinebitur penultimi termini, quem æquatio transformata habere debet; in præsenti scilicet casu $\frac{3a^3}{a} + \frac{2pa^2}{a} + \frac{qa}{a} = 3a^2 + 2pa + q =$

$\frac{m.\overline{m-1}}{1.\quad 2} a^2 + \overline{m-1}\, pa + q$.

III. Si penultimus iste coëfficiens multiplicetur ut supra per exponentes quantitatis a divisos per $2a$, habebitur *coëfficiens* termini antepenultimi; videlicet

cet hoc in casu $\frac{6a^2}{2a} + \frac{2pa}{2a} = 3a + p = ma + p$.

IV. Si coëfficiens iste multiplicetur ut ante per exponentes quantitatis a divisos per $3a$, orietur coëfficiens termini, qui antepenultimum præcedit, & sic deinceps.

Methodus igitur facillima hinc elicitur ad æquationem datam in aliam expedite transformandam, in qua datæ radices sint quantitate data auctæ, vel imminutæ.

Exemplum. Proponatur æquatio $x^3 + px^2 + qx + r = 0$ in aliam transformanda, in qua sit $x = y + a$. Pro x scribo a, & pro ultimo transformatæ termino statim habeo $a^3 + pa^2 + qa + r$. Hinc termini penultimi coëfficiens in transformata est $\frac{3a^3 + 2pa^2 + qa}{a} = 3a^2 + 2pa + q$, adeoque penultimus transformatæ terminus $3a^2 y + 2pay + qy$. Pro coëfficiente termini antepenultimi habeo $\frac{6a^2 + 2pa}{2a} = 3a + p$; quare pro termino antepenultimo stabit $3ay^2 + py^2$; ac tandem pro primo termino $\frac{3ay^3}{3a} = y^3$; quocirca æquatio transformata erit

$y^3 +$

$$\left.\begin{array}{l} y^3+3ay^2+3a^2y+a^3 \\ +\ py^2+2pay+pa^2 \\ \quad +\ qy+qa \\ \qquad +\ r \end{array}\right\} = 0$$

quare omnibus æquationibus tertii gradus transformandis inservit. Esto namque $x^3 - x^2 - 14x + 24 = 0$, cujus radices $2, 3, -4$, in aliam transformanda, in qua $x + a = y$, seu $x = y - a$; quoniam $p = -1$, $q = -14$, $r = 24$, & a est quantitas negativa; transformata substituendo fiet

$$\left.\begin{array}{l} y^3 - 3ay^2 + 3a^2y - a^3 \\ -\ y^2 + 2ay - a^2 \\ \quad -\ 14y + 14a \\ \qquad +\ 24 \end{array}\right\} = 0$$

ubi si statuatur $a = 4$, æquatio transformata, in qua $y - 4 = x$, seu $y = x + 4$, evadet $y^3 - 13y^2 + 42y = 0$, adeoque ejus radices erunt $2+4$, $3+4$, $-4+4$, sive $6, 7, 0$, fit enim secundi gradus æquatio $y^2 - 13y + 42 = 0$ (374.).

446. *Scholion.* Ut hæc omnia distinctius inspiciantur, æquationem transformatam inverso terminorum ordine, (seu sumendo a pro primo termino, ut sit $x = a + y$) sic expono

$$\begin{array}{llll} a^m & +\ ma^{m-1}y & +\ m\left(\frac{m-1}{2}\right)a^{m-2}y^2 & \&c. \\ +\ pa^{m-1} & +\ (m-1)pa^{m-2}y & +\ (m-1)\left(\frac{m-2}{2}\right)pa^{m-3}y^2 & \&c. \\ +\ qa^{m-2} & +\ (m-2)qa^{m-3}y & +\ (m-2)\left(\frac{m-3}{2}\right)qa^{m-4}y^2 & \&c. \\ +\ ra^{m-3} & +\ (m-3)ra^{m-4}y & +\ (m-3)\left(\frac{m-4}{2}\right)ra^{m-5}y^2 & \&c. \\ \&c. & \&c. & \&c. & \end{array}$$

Ubi

Ubi ſtatim apparet, coëfficientem cujuslibet termini y^r generatim erui ex coëfficiente termini præcedentis y^{r-1}, multiplicando ſingulas hujus coëfficientis portiones ſignis interjectas per indices reſpondentes ipſius a, & dividendo productum per ra. Sic primus terminus $a^m + pa^{m-1} + qa^{m-2} + ra^{m-3} +$ &c. ſtatim habetur ſufficiendo a pro x in propoſita, & hic eſt ultimus transformatæ terminus. Secundi termini coëfficiens

$ma^{m-1} + (m-1)pa^{m-2} + (m-2)qa^{m-3} + (m-3)ra^{m-4} +$ &c. oritur ex termino præcedente multiplicando ſingulas ejus partes ſignis interjectas per indices reſpondentes ipſius a, & dividendo productum per a; & hic eſt coëfficiens termini penultimi in transformata.

Tertii termini coëfficiens exurgit ex coëfficiente ſecundi, multiplicando ut ante ſingulas ejus partes per indices reſpondentes, & dividendo per $2a$; & hic eſt terminus antepenultimus in transformata; & ſic deinceps.

447. *Coroll.* 1. Si æquationis datæ radices augeantur, veræ augentur, & falſæ non modo imminuuntur, ſed nonnunquam in veras abeunt. Sic ſi radix falſa -3 augeatur numero $+5$, fit $-3+5=2$.

448. *Coroll.* 2. Si æquationis datæ radices minuantur, falſæ augentur, at veræ minuuntur, & aliquando tranſeunt in falſas. Sic ſi radix vera $+4$ minuatur numero 7 fit $+4-7=-3$.

449. *Coroll.* 3. Augendo igitur æquationis datæ radices quantitate, quæ sit major maxima radice falsa, omnes radices falsæ migrant in veras; & contra minuendo æquationis radices quantitate, quæ sit major maxima radice vera, omnes radices veræ evadunt falsæ; quare minuendo radices quantitate, quæ sit minor quam minima radix positiva, necnon augendo radices quantitate, quæ sit minor numero, quem proferi minima radix negativa, radicum signa eadem in transformata persistunt, quæ in proposita.

450. *Coroll.* 4. Hinc si omnia novæ æquationis signa per vices procedant, nempe si omnes radices sint veræ (392.), indicio est, quantitatem adsumptam majorem esse quavis radice falsa æquationis propositæ, eas enim vertit in veras. At si novæ æquationis signa sint omnia similia, ac proinde omnes radices falsæ, indicio est, omnes æquationis datæ radices veras imminutas esse quantitate ipsis majore, quæ idcirco eas in falsas commutavit (*loc. cit.*).

451. *Coroll.* 5. Si radices veras aut falsas augendo vel minuendo, nova æquatio una dimensione deprimitur (ut fieri vidimus in æquatione $y^3 - 13y^2 + 42y * = 0$, quæ per y divisa evadit $y^2 - 13y + 42 = 0$), quantitas aucta, vel imminuta est una ex radicibus æquationis propositæ.

452. Propositio I.I. *Æquationis radices per datam quantitatem multiplicare, ac dividere.*

Resolutio. Esto $x^m + px^{m-1} + qx^{m-2} + rx^{m-3} +$ &c. $= 0$, cujus radices x sint per a multiplicandæ. Fiat

an

$an = y$, erit $n = \frac{y}{a}$; $n^m = \frac{y^m}{a^m}$; $pn^{m-1} = \frac{py^{m-1}}{a^{m-1}}$;

$qn^{m-2} = \frac{qy^{m-2}}{a^{m-2}}$; $rn^{m-3} = \frac{ry^{m-3}}{a^{m-3}}$ &c. quare subrogando prodibit

$$\frac{y^m}{a^m} + \frac{py^{m-1}}{a^{m-1}} + \frac{qy^{m-2}}{a^{m-2}} + \frac{ry^{m-3}}{a^{m-3}} + \&c. = 0 .$$

Et multiplicando per a^m, $y^m + pay^{m-1} + qa^2y^{m-2} + ra^3y^{m-3} + \&c. = 0$. *Quod erat primum.*

Quod si ponatur $a = \frac{1}{c}$, multiplicatio vertitur in divisionem, fitque nova æquatio

$$y^m + \frac{py^{m-1}}{c} + \frac{qy^{m-2}}{c^2} + \frac{ry^{m-3}}{c^3} + \&c. = 0.$$ *Quod erat secundum.*

453. *Coroll.* 1. Ergo ut æquationis propositæ radices x per datam quantitatem a multiplicentur, aut dividantur, posito in primo casu $ax = y$, seu $x = \frac{y}{a}$, fiat, sumpto denominatore a, progressio geometrica $1, a, a^2, a^3, a^4$, &c. collocanda ordinatim sub terminis omnibus datæ æquationis, incipiendo a primo ejusdem termino; deinde per singulos hujusce progres-

gressionis terminos multiplicentur æquationis propositæ respectivi coëfficientes, retentis eorum signis; tum pro x numerator y substituatur, idemque præstetur, si fiat $\frac{1}{x} = y$, vel $x = \frac{1}{y}$. In altero casu, posito $\frac{x}{a} = y$, vel $x = ay$, eadem peragantur hoc tantum cum discrimine, quod pro multiplicatione divisio per a instituatur. Sic si æquationis $x^4 - 4x^3 + 5x^2 - 6x + 4 = 0$ radices sint per 3 multiplicandæ, hoc est si statuatur $3x = y$, vel $x = \frac{y}{3}$, ita erit procedendum

$$x^4 - 4x^3 + 5x^2 - 6x + 4 = 0$$
$$1,\ 3,\ 9,\ 27,\ 81$$
$$\overline{y^4 - 12y^3 + 45y^2 - 162y + 324 = 0}$$

At si æquatio $x^3 * - 36x - 54 = 0$ sit per 3 dividenda, fiet

$$x^3 * - 36x - 54 = 0$$
$$1,\ 3,\ 9,\ 27$$
$$\overline{y^3 * - 4y - 2 = 0}, \text{ ubi } y = \tfrac{1}{3} x.$$

454. *Coroll.* 2. Utilis interdum evadit hæc methodus, dum æquationes a radicalibus liberare cupimus, ipsas per progressionem geometricam ex radicalibus concinnatam multiplicando, aut dividendo, ut sequentia indicant exempla

I.

I. $x^3 + 2x^2\sqrt{a} - 15x + 9\sqrt{a} = 0$

$1\ ,\ \sqrt{a}\ ,\ a\ ,\ a\sqrt{a}$

Multiplicando $y^3 + 2ay^2 - 15ay + 9a^2 = 0$

Dividendo $y^3 + 2y^2 - \frac{15y}{a} + \frac{9}{a} = 0$ hic $y = \frac{x}{\sqrt{a}}$.

II. $x^3 - ax^2\sqrt[3]{2} + abx\sqrt[3]{32} - a^2b$

$1\ ,\ \sqrt[3]{2}\ ,\ \sqrt[3]{4}\ ,\ 2$

Dividendo $y^3 - ay^2 + 2aby - \frac{1}{2}a^2b$; hic $y = \frac{x}{\sqrt[3]{2}}$.

Verum id sic assequimur, cum radicales homogenei per vices, a secundo termino incipiendo, in æquatione procedunt, nec refert, terminum aliquem in locis paribus deficere.

455. *Coroll.* 3. Inveniri etiam potest æquatio, in qua termini alicujus cœfficiens minor sit cœfficiente termini homogenei æquationis datæ. Sit ex. gr. æquatio $x^3 - 40x^2 + 48x - 100 = 0$, & alia expetatur, cujus secundus terminus sit minor termino $40x^2$. Quoniam terminus iste dividitur per 4, fiat series geometrica ab unitate incipiens, cujus secundus terminus sit 4, & per hanc omnes termini dividantur; nempe

$x^3 - 40x^2 + 48x - 100 = 0$

$1\ ,\ 4\ ,\ 16\ ,\ 64$

$y^3 - 10y^2 + 3y - \frac{25}{16} = 0$, in qua $y = \frac{1}{4}x$.

 Si

Si tertii termini coëfficiens 48 imminui in nova æquatione debeat, quoniam hunc numerus 16 metitur, formetur memorata series, cujus tertius terminus incipiendo ab unitate sit 16, factaque per illam divisione, æquatio exurget quæsita cum ultima congrens, cujusque radices sunt ut ante pars quarta radicum datæ æquationis.

456. *Coroll.* 4. Per multiplicationem ac divisionem æquatio data ad simpliciorem formam reduci aliquando potest; en Exempla

$$\text{I. } x^3 - 4x^2 + 5x - \frac{1}{8} = 0$$

$$\underline{1,\ 2,\ 4,\ 8}$$

$$y^3 - 8y^2 + 20y - 1 = 0;\ \text{in qua } y = 2x.$$

$$\text{II. } x^3 - 18x^2 + 104x - 192 = 0$$

$$\underline{1,\ 2,\ 4,\ 8}$$

$$y^3 - 9y^2 + 26y - 24 = 0;\ \text{in qua } y = \frac{1}{2}x.$$

457. *Coroll.* 5. Si æquationis ordinatæ primum terminum aliquo coëfficiente affectum, ad alium in unitatem ductum reducere lubeat, coëfficiens iste ponatur pro secundo termino progressionis geometricæ ab unitate incipientis, tum hac progressione sub terminis propositæ de more exarata, fiat primum multiplicatio, deinde coëfficientis divisio, nimirum

$7x^3 -$

$7x^3 - 2x^2 - 5x + 3 = 0$
$\underline{1 \quad 7 \quad 49 \quad 343}$
$7y^3 - 14y^2 - 245y + 1029 = 0$; & dividendo per 7;
$y^3 - 2y - 35y + 147 = 0$

458. *Coroll.* 6. Possumus itidem æquationem a fractionibus liberare, multiplicando scilicet radicem æquationis per omnium denominatorum productum, aut per numerum denominatores omnes metientem. Pergo ad exempla.

I. $x^3 - \frac{px^2}{a} + \frac{qx}{b} - \frac{r}{c} = 0$

$\underline{1 \;,\; abc \;,\; a^2b^2c^2 \;,\; a^3b^3c^3}$
$y^3 - bcpy^2 + a^2bc^2qy - a^3b^3c^2r = 0$

In hac transformata habetur $y = abcx$, adeoque $x = \frac{y}{abc}$. Si æquationis datæ termini sint partim integri, partim fracti, integri ad fractos, facta per unitatem divisione, reducantur. Præstat autem interdum multiplicationem per communem denominatorum minorem, si detur, instituere, nimirum

II. $xx * + \frac{2}{3}x - \frac{81}{27} = 0$

$\underline{1 \;,\; 3 \;,\; 9 \;,\; 27}$
$y^2 * + 6y - 81 = 0$; ubi $y = 3x$.

Coëfficiens pariter, quem habet maxima incognitæ potestas, ad unitatem, si lubeat, redigi facile potest; nam facta per eundem coëfficientem divisione, æqua-

 tio

tio a fractionibus liberanda huc pertinet; sit enim $ax^3 - px^2 + qx - r = 0$; dividendo per a, fit $x^3 -$

$\frac{px^2}{a} + \frac{qx}{a} - \frac{r}{a}$; Hinc

$$\text{III. } x^3 - \frac{px^2}{a} + \frac{qx}{a} - \frac{r}{a} = 0$$

$$\frac{\quad 1 \;,\quad a \;,\quad a^2 \;,\quad a^3 \quad}{y^3 - py^2 + aqy - a^2r = 0;}$$ ubi $y = ax$.

459. *Coroll.* 7. Si novam invenire oporteat æquationem, cujus radices per propositæ radices multiplicatæ faciant quantitatem datam, hoc ita obtinebimus. Esto $x^3 - px^2 + qx - r = 0$, & a quantitas data. Pone $xy = a$; erit $x = \frac{a}{y}$; quare (403.)

$$\frac{\begin{array}{c} x^3 - px^2 + qx - r = 0 \\ 1 \;,\; y \;,\; y^2 \;,\; y^3 \end{array}}{a^3 - pa^2y + qay^2 - ry^3 = 0,}$$

mutatis vero signis, ac dividendo per r, provenit

$y^3 - \frac{qay^2}{r} + \frac{pa^2y}{r} - \frac{a^3}{r} = 0$; in qua $y = \frac{a}{x}$.

Pari methodo invenitur æquatio, cujus radices æquationis propositæ radicibus sint reciprocæ; si enim fiat $a = 1$, erit $\frac{1}{x} = y$, & $x = \frac{1}{y}$, quamobrem si proposita fuerit $x^3 - px^2 + qx - r = 0$, ejus reci-

pro-

proca erit $1 - py + qy^2 - ry^3 = 0$, seu $y^3 - \frac{qy^2}{r} + \frac{py}{r} - \frac{1}{r} = 0$, ubi (1°.) coëfficientium ordo invertitur, quapropter si secundus terminus in proposita defuisset, tertius in transformata evanuisset, & vicissim; (2.°) maxima æquationis datæ radix evadit minima in transformata.

460. *Coroll.* 8. Transformari quoque data æquatio potest in aliam, in qua coëfficiens cujuslibet termini, ultimo haud excepto, datæ quantitati fiat æqualis, idque inveniendo quantitatem, per quam æquationis datæ radices multiplicentur. At quomodo quantitas ista inveniatur, ex sequentibus patebit.

Esto proposita $x^3 - px^2 + qx - r = 0$; quantitas data $= m$, quæsita $= z$. Fiat $x = \frac{y}{z}$; ergo

$$\begin{array}{c} x^3 - px^2 + qx - r = 0 \\ 1 \;,\; z \;,\; z^2 \;,\; z^3 \\ \hline y^3 - pzy^2 + qz^2y - rz^3 = 0. \end{array}$$

Jam vero si secundi termini coëfficiens p fieri debeat $= m$, ponatur $pz = m$; erit $z = \frac{m}{p}$. Si tertii termini coëfficiens $q = m$, ponatur $qz^2 = m$; erit $z^2 = \frac{m}{q}$; $z = \sqrt{\frac{m}{q}}$. Si terminus ultimus $rz^3 = m$;

erit

erit $z^3 = \frac{m}{r}$; $z = \sqrt[3]{\frac{m}{r}}$. Itaque si secundus æquationis terminus ducatur in valorem inventum ipsius z, aut tertius in quadratum, aut quartus in cubum ejusdem valoris, & ita porro, jussa exurget æquatio.

Exemplum I. Sit æquatio $x^3 - 12x^2 + 9x - 27 = 0$, & alia loco ipsius inveniri debeat, in qua coëfficiens 12 secundi termini sit 2; erit $p = 12$, $m = 2$, $z = \frac{m}{p} = \frac{2}{12} = \frac{1}{6}$. Data igitur æquatio multiplicetur per $\frac{1}{6}$, videlicet

$$x^3 - 12x^2 + 9x - 27 = 0$$
$$1, \quad \frac{1}{6}, \quad \frac{1}{36}, \quad \frac{1}{216}$$

$$y^3 - 2y^2 + \frac{1}{4}y - \frac{1}{8} = 0;$$ in qua $y = \frac{1}{6}$;

Exemplum II. Si loco propositæ æquationis alia exoptetur, in qua tertii termini coëfficiens sit 4, erit $q = 9$, $m = 4$, $z = \sqrt{\frac{m}{q}} = \sqrt{\frac{4}{9}} = \frac{2}{3}$; adeoque multiplicatis propositæ radicibus per $\frac{2}{3}$, eruetur

x^3

$$x^3 - 12x^2 + 9x - 27 = 0$$

$$1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}$$

$$y^3 - 8y^2 + 4y - 8 = 0; \text{ in qua } y = \frac{2}{3} x.$$

Exemplum III. Si pro data æquatione alia quæratur, ubi quarti termini coëfficiens sit $= 1$; erit $r = 27$, $m = 1$, $z = \sqrt[3]{\frac{1}{27}} = \frac{1}{3}$; quapropter

$$x^3 - 12x^2 + 9x - 27 = 0$$

$$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$$

$$y^3 - 4y^2 + y - 1 = 0; \text{ in qua } y = \frac{1}{3} x.$$

461. Propositio IV. *Æquationem radices reales negativas vel solas, vel cum realibus positivis commixtas habentem, in aliam positivis omnibus prædictam transformare.*

Resolutio. Si omnes propositæ termini sint positivi, mutando signa in locis paribus, solvitur problema. Sin minus, maximo cöefficiente negativo (ne ultimo quidem termino excepto) posito $= -m$, fiat $\overline{m+1} - y = x$, ac substituendo, nova exurget æquatio quæsita.

Dem.

Dem. Esto $x^3 - mxx - nx - r = 0$, ubi omnes termini, primo excepto, sint negativi, sitque $-m$ maximus coëfficiens negativus; substitue $\overline{m+1} - y$ pro x in æquatione, & prodibit

$$x^3 = \overline{m+1}^3 - 3.\overline{m+1}^2 y + 3.\overline{m+1}y^2 - y^3$$

$$-mx^2 = -m.\overline{m+1}^2 + 2m.\overline{m+1}y - m.y^2$$

$$-nx = -n.\overline{m+1} + ny$$

$$-r = -r$$

Primum transformatæ terminum intuenti constabit inito calculo, esse $\overline{m+1}^3 - m.\overline{m+1}^2 = \overline{m+1}^2$; quare ex $\overline{m+1}^2$ sublato $n.\overline{m+1}$, residuum non erit minus quam $m+1$, quod (*ex hyp.*) $n < m$. Sed si ex $m+1$ dematur $r < m$, superest quantitas positiva; ergo primus transformatæ terminus est positivus. Si in secundo termino ex $-3.\overline{m+1}.\overline{m+1}y$ subtrahatur $2m.\overline{m+1}y$, residuum non erit minus quam $-\overline{m+1}y$, ex quo si tollas $+ny$, remanebit quantitas negativa.

In tertio termino patet, esse $3.\overline{m+1}y^2 > my^2$, adeoque quantitas est positiva;

Er-

Ergo in transformata termini cum signis alternis se consequuntur, ut accidit in terminis quos habet quantitatis $\overline{m+1-y}$ potestas quælibet.

Si $-n$, vel $-r$ sit maximus coëfficiens negativus, eadem est demonstratio; quod ratiocinium æquationi cujuscumque gradus applicari potest, in qua omnes termini, demptor primo, sint negativi.

Sed quia termini cujuslibet potestatis $(\overline{m+1-y})^p$ signa habere debent (65. n. IV.) cum vicissitudine $+$ & $-$; apertum est, coëfficientes positivos, qui terminos illos multiplicant, mutare non posse hanc vicissitudinem in transformata, & id tantum efficere coëfficientes negativos; ergo quum in proposita omnes coëfficientes adsumpti sint negativi, demonstratio casibus omnibus possibilibus satisfacit. *Quod erat* &c.

Exemplum. Esto $x^3-2x^2+3x+6=0$. Maximo coëfficienti negativo -2 reddito positivo addatur unitas; fiet $3-y=x$, quare substituendo, & summando habetur æquatio quæsita $0=24-18y+7y^2-y^3$.

462. *Corell.* 1. Si primus propositæ terminus non habeat unitatem pro coëfficiente, velut $ax^3-mx^2-nx-r=0$, sitque n maximus coëfficiens negativus, sumatur $\frac{n}{a}+1-y=x$, vel $\frac{n+a}{a}-y=x$, & substituendo, exurget transformata, cujus signa vi ejusdem ratiocinii per vices se succedere demonstrabitur.

463.

463. *Corollar.* 2. Per substitutionem quantitatis $\overline{n+1}-y$ pro x in proposita, radices negativæ evadunt positivæ in transformata, & singulæ augentur quantitate positiva $n+1$; quæ vero erant in proposita positivæ, evadunt negativæ in transformata, sed imminuuntur eadem quantitate $n+1$. Ut id clarius intelligatur, sint $x-a=0$, $x+b=0$ æquationes simplices æquationis propositæ, & $n+1=f$. Substitue $f-y$ pro x. Æquationis transformatæ æquationes simplices erunt $f-a-y=0$, $f+b-y=0$. Ex prima habetur $y=-a+f$, ubi radix a, quæ in $x-a=0$, vel in $x=a$ erat positiva, evadit negativa, sed imminuta quantitate positiva f; ex altera habetur $y=b+f$, ubi radix b, quæ in $x+b=0$, vel in $x=-b$ erat negativa, fit positiva, sed aucta quantitate f. Hinc ut omnes radices in transformata emergant positivæ, necesse est, ut $n+1=f$, seu maximus coëfficiens negativus affirmative acceptus, & auctus unitate, sit major quam a, adeoque major omnibus propositæ radicibus positivis.

464. *Coroll.* 3. Contra si æquatio data sit in alteram transformanda, in qua omnes radices sint negativæ, sive omnes termini sint positivi; peracta dicta operatione, signa in transformatæ terminis paribus mutentur (443.). Vel brevius, pro x substitue $\overline{m+1}+y$, ac terminorum in transformata coëfficientes omnes prostabunt positivi, ideoque $m+1$ erit rursus major maxima propositæ radice positiva. Quum enim sit $x=a=f+y$; $x=-b=f+y$; erit $y=a-f$; $y=$

$y = -b-f$; ergo ut omnes transformatæ radices y evadant negativæ, esse debet $f = m + 1 > a$.

465. *Coroll.* 4. Hinc si in proposita mutentur signa in locis paribus, maximus coëfficiens negativus affirmative acceptus erit major maxima radice positiva; sed hæc ante signorum mutationem erat negativa; ergo sciri quoque facile potest quantitas, quæ sit major maxima radice negativa quoad numerum, cui signum − præfigitur. Sic in proposita $x^3 - m^2 - nx - r = 0$, si mutentur signa in locis paribus, fiet $x^3 + mx^2 - nx + r = 0$, ac proinde $n + 1$ erit major maxima propositæ radice negativa.

466. *Corollarium* 5. Reciprocè si statuendo $x = \frac{1}{y}$, æquatio proposita ordinetur per y (402.), deinde per dictam methodum indagetur radix major maxima transformatæ radice; hæc (ob $y = \frac{1}{x}$) erit, ut patet, minor minima propositæ radice; adeoque agendo ut in duobus postremis Corollariis præcedentibus, haberi etiam potest radix minor minima propositæ radice tam positiva, quam negativa.

467. *Scholion.* Quamquam maximus æquationis cujuscumque coëfficiens negativus unitate auctus est semper limes superans singulas omnes æquationis radices positivas, attamen raro est limes propinquus; quare ut talis habeatur, in æquatione $x^m + px^{m-1}$ +

$+ qx^{m-2} + rx^{m-3} +$ &c. ſtatue $x - a = y$, ſeu $x = y + a$. Vide alibi (445.) transformatam hinc emergentem. Ergo ſi quantitas a talis ſit, ut omnes transformatæ coëfficientes evadant poſitivi, quum nulla ſignorum tunc occurrat variatio, omnes ipſius y valores erunt negativi, adeoque ob quantitatem $x - a$ negativam erit $a > x$. Quoniam igitur a ſuperat in hac hypotheſi maximam radicem poſitivam ejuſdem x, erit a limes æquationis propoſitæ. Ergo ſi pro a in transformata ſubſtituamus x, coëfficientes ex dictis (446.) evadent

$$x^m + px^{m-1} + qx^{m-2} + rx^{m-3} + \text{\&c.}$$

$$mx^{m-1} + (m-1)px^{m-2} + (m-2)qx^{m-3} + (m-3)rx^{m-4} + \text{\&c.}$$

$$\frac{m(m-1)}{2}x^{m-2} + (m-1)\frac{(m-2)}{2}px^{m-3} + (m-2)\frac{(m-3)}{2}qx^{m-4} + \text{\&c.}$$

&c. &c.

Hinc ut quantitatem habeamus, quæ maximam propoſitæ radicem propiùs ſuperet, quantitas minima tentando eſt invenienda, quæ expoſitas expreſſiones in caſibus particularibus reddat omnes affirmativas.

Quod ſi limites minimæ radicis poſitivæ, maximæque, vel minimæ radicis negativæ cupiantur, operandum, ut in Corollariis præcedentibus; ubi advertendum, regulam iſtam etiam radicibus ipſis rationalibus inveniendis inſervire.

Exemplo ſit $x^4 - 2x^3 - 25x^2 + 26x - 120 = 0$; ex qua ſit

$x^4 -$

$x^4 - 2x^3 - 25x^2 + 26x - 120$
$2x^3 - 3x^2 - 25x + 13$
$6x^2 - 6x - 25$
$2x - 1$

Minima quantitas, quæ omnes iſtas expreſſiones reddit poſitivas, eſt ipſa radix 6, quæ propoſitam ſimul reducit ad zero. Quum autem una radix adſit negativa in propoſita, mutentur ſigna in locis paribus, ac proveniet

$x^4 + 2x^3 - 25x^2 - 26x - 120$
$2x^3 + 3x^2 - 25x - 13$
$6x^2 + 6x - 25$
$2x + 1$

Minima quantitas, de qua eſt quæſtio, reperietur eſſe ipſa radix 5, adeoque radix negativa propoſitæ eſt -5.

Similiter in æquatione $x^5 - 2x^4 - 10x^3 + 30x^2 + 63x + 120 = 0$ minima quantitas poſitiva quæſito ſatisfaciens reperietur $= 2$, minima negativa (conſiderando numerum, cui ſignum $-$ præfigitur) $= -3$, adeoque quinque æquationis radices inter -3, & 2 conſiſtent.

468. Propoſitio V. *Propoſitam æquationem in aliam transformare, cujus radices adæquent poteſtates radicum, queis propoſita conſtat.*

Reſolutio. Propoſita æquatio ordine inverſo exhibita ſit $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7 + ix^8 +$ &c. $= 0$. Ut transformatæ quælibet radix z adæquet quadratum xx, colloca z ubi xx adeſt in propoſita, ac fiet $a + bx + cz + dxz + ezz + fxzz + gz^3 + hxz^3 + iz^4 +$ &c. $= 0$.

 Po-

Pone $a + cz + ezz + gz^3 + \&c. = P$,

& $b + dz + fzz + \&c. = Q$;

proposita evadet $P + Qx = 0$. Hanc, ut x eliminetur, multiplica per x, scribendo z loco xx, & habebis $Px + Qz = 0$, seu $x = -\frac{Qz}{P}$, quo valore in æquatione $P + Qx = 0$ subrogato, prodit $PP - QQz = 0$. Substitue, atque obtinebis transformatam primo quæsitam.

Esse nunc debeat $z = x^3$. Ubi reperitur x^3 in proposita substitue z; proveniet

$a + bx + cx^2 + dz + exz + fx^2z + gz^2 + hxz^2 + ix^2z^2 + \&c. = 0 \ldots . (A)$.

Statue $a + dz + gzz + \&c. = P$

$b + ez + hzz + \&c. = Q$

$c + fz + izz + \&c. = R$

Æquatio (A) evadet $P + Qx + Rx^2 = 0 \ldots . (B)$

Hæc multiplicata per x fit

$Px + Qx^2 + Rx^3 = Rz + Px + Qx^2 = 0 \ldots . (C)$

quæ rursus multiplicata per x evadit

$Rxz + Px^2 + Qx^3 = Qz + Rxz + Px^2 = 0 \ldots . (D)$

Duc tres istas æquationes (B), (C), (D), primam in L, secundam in M, tertiam in N, & obtinebis

$$LP + LQx + LRx^2 = 0$$
$$MRz + MPx + MQx^2 = 0$$
$$NQz + NRxz + NPx^2 = 0.$$

Ad x & xx eliminandas fac $LQ + MP + NRz = 0 \ldots . (E)$ $LR + MQ + NP = 0 \ldots . (F)$; & proposita migrabit in $LP + MRz + NQz = 0 \ldots . (G)$.

Ex

Ex æquationibus (E), (F) colligitur $L = -\frac{MP - NRz}{Q} = -\frac{MQ - NP}{R}$, ac proinde $MPR + NRRz = MQQ + NPQ$, seu $MPR - MQQ = NPQ - NRRz$, adeoque $\frac{M}{N} = \frac{PQ - RRz}{PR - QQ}$. Si igitur ponatur $M = PQ - RRz$, erit $N = PR - QQ$, ac proinde $L = RQz - PP$.

Hisce valoribus in æquatione (G) substitutis, exurget $3PQRz - P^3 - Q^3z - R^3zz = 0$; quæ est transformata secundo expetita.

Methodus consimilis pro altioribus potestatibus est adhibenda. *Quod erat* &c.

469. Propositio VI. *Si æquatio quælibet* (A) *radices omnes reales continens per progressionem arithmeticam* (B) *multiplicetur*, *prove-*

(A)	$x^n + px^{n-1} + qx^{n-2} + rx^{n-3} \ldots + T = 0$
(B)	$n,\ n-1,\ n-2,\ n-3,\ n-n$

niet $nx^n + \overline{n-1}\,px^{n-1} + \overline{n-2}\,qx^{n-2} + \overline{n-3}\,rx^{n-3}$ + &c. $= 0$, *seu dividendo per* x, *quod homogeneus comparationis deficit*, *fiet* (C) $nx^{n-1} + \overline{n-1}\,px^{n-2}$

 +

$+\overline{n-2}\,qx^{n-3}+\overline{n-3}\,rx^{n-4}\ldots\ldots+S=0$. *Dico hujus quoque æquationis* (C) *radices omnes esse reales.*

Præparatio. Fiat $x=y+t$; ultimus transformatæ terminus erit (*D*) $t^n+pt^{n-1}+qt^{n-2}+rt^{n-3}$ $\ldots\ldots+R$ (396. n. I.); penultimus (*E*) $y(nt^{n-1}+\overline{n-1}\,pt^{n-2}+\overline{n-2}\,qt^{n-3}+\&c.)$ (*Ibid.* n. V.). Sed si pro t ponatur x, sive una ex propositæ radicibus, terminus iste ultimus (*D*) evanescet, quod evanescit proposita; ergo in hac hypothesi remanet penultimus (*E*), qui per y divisus evadit

(*F*) $nt^{n-1}+\overline{n-1}\,pt^{n-2}+\overline{n-2}\,qt^{n-3}\ldots+Q=0$, quæ æquatio, si fiat $t=x$, eadem est cum æquatione (*C*). Oportet igitur demonstrare, hujus æquationis (*F*) radices omnes esse reales.

Demonstratio. Æquationis itaque propositæ factores sint $x-a$, $x-b$, $x-c$, &c. nec non $x+f$, $x+g$, $x+h$, &c. hique ita dispositi concipiantur, ut radices positivæ a, b, c, &c. ex una parte, & negativæ f, g, h, &c. ex altera separatæ ex minoribus ad majores procedant, nulla habita ratione signorum. Si fiat, ut supra, $x=y+t$, transformatæ factores erunt $y+t-a$, $y+t-b$, $y+t-c$, &c. $y+t+f$, $y+t+g$, $y+t+h$, &c. Sufficiatur pro t prima propositæ radix a; transformatæ factores erunt $y+a-a$, $y+a-b$, $y+a-c$, &c. $y+a+f$, $y+a+g$, $y+a+h$, &c. Quoniam vero propter $y+a-a=0$, ul-

ultimus transformatæ terminus, ut diximus, evanescit, factores transformatæ ultimo termino carentis, quam *mutilatam* vocabimus, erunt $y+a-b, y+a-c$, &c. $y+a+f, y+a+g, y+a+h$, &c. Ponatur jam pro x secunda radix b; transformatæ integræ factores erunt $y+b-a, y+b-b, y+b-c$, &c. $y+b+f, y+b+g, y+b+h$, &c. adeoque mutilatæ factores $y+b-a, y+b-c$, &c. $y+b+f, y+b+g$, &c. Similiter si tertia radix c pro x subrogetur, transformatæ mutilatæ factores reperientur $y+c-a, y+c-b, y+c-d$, &c. $y+c+f, y+c+g$, &c. atque ita continuetur usque ad maximam radicem positivam.

At quia (*ex hyp.*) $a<b, b<c, c<d$, &c. radices $a-b, a-c$, &c. quæ ultimum transformatæ mutilatæ terminum (E), vel (F) multiplicant, sunt omnes negativæ, reliquis $a+f, a+g$, &c. positivis manentibus; ex substitutione autem b pro x, radices $b-a, b-c$, &c. sunt omnes præter primam negativæ, ceteris $b+f, b+g$, &c. stantibus positivis; reliquum est, ut ultimus mutilatæ terminus (F) in substitutione a pro x signum habere debeat diversum ab illo, quod habet in substitutione b pro x. Pari methodo invenietur, ultimum mutilatæ terminum signum habere in substitutione b pro x diversum ab illo, quod habet in substitutione c pro x, quia tunc in transformata mutilata radices forent $c-a, c-b, c-d$, &c. quæ præter primam & secundam sunt omnes negativæ; & sic deinceps. Si præterea pro x substituere ut supra successive lubeat singulas radices negativas $-f, -g, -h$, &c. patet, omnia

 signa

ſigna in dicto termino (F) mutari debere, adeoque hoc etiam in caſu perinde atque in allato ſigna ex poſitivis ad negativa, & a negativis ad poſitiva, & ita porro, vel viciſſim, ſemper tranſibunt. Sed cum id accidit, omnes hujus termini (F) vel (C) poſiti $= 0$ radices permanent ſemper reales (429.); ergo &c. *Quod erat* &c.

470. *Scholion* I. Quæcumque ſit progreſſio arithmetica ab enunciata (B) diverſa propoſitam (A) multiplicans, radices omnes in producto manebunt itidem reales. Multiplicetur enim propoſita $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} \ldots + T = 0$ per progreſſionem arithmeticam generalem $m + kn, m + k(n-1), m + k(n-2)$, &c. proveniet $m(x^n + px^{n-1} + qx^{n-2} + rx^{n-3} + \&c.) + kx(nx^{n-1} + \overline{n-1}\,px^{n-2} + \overline{n-2}\,qx^{n-3} + \&c. + R)$. Sed $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} + \&c. = 0$; ergo ſupereſt productum ex kx in quantitatem $nx^{n-1} + \overline{n-1}\,px^{n-2} + \overline{n-2}\,qx^{n-3} + \&c.$ quæ poſita $= 0$, & per kx diviſa eſt eadem cum recenſita (F), vel (C), cujuſque idcirco radices (469.) perſiſtunt omnes reales. Sic ſi ſtatuatur $m = 0$, $n = 0$, $k = -1$, dicta progreſſio $m + kn$, $m + k(n-1)$, $m + k(n-2)$, &c. evadet $0, 1, 2, 3$, &c. ſin autem ponatur $m = 0, k = 1$, fiet $n, n-1, n-2,$

$n-2$, &c. hoc est æquatio data per incognitæ exponentes, ut egimus, multiplicabitur.

471. *Coroll.* 1. Si propositæ per progressionem quamlibet arithmeticam multiplicatæ productum rursus per aliam quamcumque progressionem consimilem multiplicetur, novumque productum fiat $= 0$, hujusmodi æquationis radices erunt iterum reales, & ita semper.

472. *Coroll.* 2. Si detur æquatio $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} \ldots + T = 0$, cujus radices sint a, b, c, d, &c. radices istæ erunt limites æquationis $nx^{n-1} + (n-1)px^{n-2} + (n-2)qx^{n-3} +$ &c. $= 0$ (419.). Ex quo colligitur, quod si a', b', c', &c. sint radices æquationis $nx^{n-1} + (n-1)px^{n-2} + (n-2)qx^{n-3}$ + &c. $= 0$, etiam illæ erunt vicissim limites primæ æquationis $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} +$ &c. $= 0$. Si enim a, b sunt limites radicis a'; & b, c limites radicis b'; cadet b inter a', & b'. Eadem ratione c cadet inter b', & c', & sic de ceteris gradatim procedendo. Jure igitur æquatio (C), & cum ea singulæ subsequentes, *æquationes limitum* appellantur.

473. *Coroll.* 3. Penultimus igitur transformatæ terminus primam exhibet æquationem limitum pro æquationis datæ limitibus determinandis.

474. *Coroll.* 4. Hinc si æquatio data habeat numerum m radicum invicem æqualium; numerus $m-1$ ra-

radicum in prima æquatione limitum consistet inter radices illas æquales, quare si quis limes in æquationem datam introductus illam reddit nihilo æqualem, non solum hic limes radicem offert communem cum æquatione data, sed simul ostendit, duas saltem in data inesse radices ipsi limiti, nec non invicem æquales. Secunda æquatio limitum habebit numerum radicum $m-2$ ex æqualibus in data existentibus, & ita porro. Sic si proponatur $x^3-8x^2+21x-8=0$, prima æquatio limitum erit $x^2-\frac{16}{3}x+7=0$, cujus radix $=3$. Sed substituendo 3 in data æquatione, prodit zero; ergo 3 est radix æquationibus utrisque communis, & duæ in æquatione data inerunt radices æquales, singulæ $=3$.

475. *Coroll.* 5. Præterea quum æquatio quadratica sit prima æquatio limitum æquationis cubicæ, invento limite, qui maximam radicem cubicam superet, limites pro æquationis cubicæ radicibus venandis erunt quatuor computato zero, qui minor quacumque radice haberi potest. Generatim vero si æquatio data sit dimensionis m, prima æquatio limitum præbebit limites $m+1$ pro radicibus æquationis datæ definiendis. Idem dicendum de secunda æquatione limitum quoad primam, & sic deinceps.

476. *Coroll.* 6. Quod si limes in data æquatione substitutus nec signa alterna, nec zero suppeditet, radices imaginarias æquatio data comprehendet.

477. *Scholion* II. Si æquationes limitum aliquas habeant radices imaginarias, radices quoque inerunt ima-

imaginariæ in proposita. Data enim ex. gr. æquatione $x^3 - px^2 + qx - r = 0$. cujus transformatâ (statuendo $x - v = y$, seu $y = x + v$) fit $y^3 + (3v - p)$ $y^2 + (3v^2 - 2pv + q)y + v^3 - pv^2 + qv - r = 0$, si ponas $v = x$, ultimus terminus $v^3 - pv^2 + qv - r$ evanescet; superstes igitur æquatio $y^2 + (3v - p)y + 3v^2 - 2pv + q = 0$ erit factum ex duobus valoribus remanentibus ipsius y, seu excessus duorum valorum ejusdem x supra tertium, qui positus fuerat $= v$; quare si duo isti excessus aliquam expressionem imaginariam includant, consimiles expressiones in duobus istis valoribus ejusdem x necessario delitescent, & sic de ceteris altioribus æquationibus. Quod demonstravimus de radicibus imaginariis, admitti nequit quoad radices reales, innumeri enim dantur casus, in quibus æquationes limitum radicibus gaudeant realibus, attamen æquatio proposita vel aliquas, vel interdum omnes habeat radices imaginarias. Exemplo sit æquatio limitum $3x^2 - (4p + 2r)x + p^2 + 2pr + q = 0$. Hæc omnes habebit radices reales, si fuerit $(p - r)^2 > 3q$; at æquatio princeps, unde ista ortum ducit, est $x^3 - (2p + r)x^2 + (p^2 + 2pr + q)x - r(p^2 + q) = 0$, quæ duas habet radices imaginarias $p + \sqrt{-q}$; $p - \sqrt{-q}$. Itidem alia æquatio limitum, multiplicando primam æquationem limitum per progressionem $0, -1, -2, -3$, quæ erit $(2p + r)x^2 - 2(p^2 + 2pr + q)x + 3r(p^2 + q) = 0$, omnes radices reales continebit, si fuerit $(p^2 + 2pr + q)^2 > (2p + r)(p^2 + q)3r$. Quod etiam valet de altioribus æquationibus. Sed quoad limites in præsentiarum ista sint satis.

478.

478. Prop. VII. *Si detur æquatio* $x^m + px^{m-1} + qx^{m-2} + rx^{m-3} +$ &c. $= 0$, *quæ identica ponatur cum æquatione* $x^m + fx^{m-1} + gx^{m-2} + hx^{m-3} +$ &c. $= 0$, *ubi nempe eadem adest incognita; ajo esse* $p = f, q = g, r = h$, &c.

Dem. Coëfficiens p secundi termini primæ æquationis continet summam omnium radicum sub signo contrario; coëfficiens q tertii termini continet producta omnium radicum quæ binæ sint combinatæ, &c. (413.). Idem accidit in secunda æquatione. Sed æquationes istæ sunt [*per hyp.*] identicæ, adeoque ex iisdem radicibus singulæ componuntur; ergo summa omnium radicum sub signo contrario in prima æquatur summæ omnium radicum sub signo contrario in secunda, &c. nimirum erit $p = f, q = g$, $r = h$, &c. *Quod erat* &c.

479. Propositio VIII. *Ex data æquatione terminos tollere.*

Resolutio. Esto $u^m \pm pu^{m-1} \pm qu^{m-2} \pm ru^{m-3}$ $\pm$ &c. Inveniri debet quantitas z, qua æquationis radices sic augeantur, aut minuantur, ut terminus propositus evanescat. Eapropter ponatur $u + z = y$, erit $u = y - z$, adeoque potestates resolvendo (67.), prodibit

$u^m =$

$$x^m = (y - z) = y^m - mzy^{m-1} + \frac{m(m-1)}{1 \cdot 2} z^2 y^{m-2} - \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} z^3 y^{m-3} + \&c.$$

$$\pm px^{m-1} = \pm p(y - z)^{m-1} = \pm py^{m-1} \mp (m-1)pzy^{m-2} \pm \frac{(m-1)(m-2)}{1 \cdot 2} pz^2 y^{m-3} \mp \&c.$$

$$\pm qx^{m-2} = \pm q(y - z)^{m-2} = \pm qy^{m-2} \mp (m-2)qzy^{m-3} \pm \frac{(m-2)(m-3)}{1 \cdot 2} qzy^{m-4} \quad \&c.$$

$$\pm rx^{m-3} = \pm r(y - z)^{m-3} = \pm ry^{m-3} \mp (m-3)rzy^{m-4} \quad \&c.$$

Quæ

Quare transformata rite dispofita erit

$$
\begin{array}{llll}
y^{m} - mzy^{m-1} & + \frac{m(m-1)}{1\,.\,2} z^{2} y^{m-2} & - \frac{m(m-1)(m-2)}{1\,.\,2\,.\,3} z^{3} y^{m-3} & + \&c. \\
\pm py^{m-1} & \mp (m-1)pzy^{m-2} & \pm \frac{(m-1)(m-2)}{1\,.\,2} pz^{2} y^{m-3} & \mp \&c. \\
 & \pm qy^{m-2} & \mp (m-2)qzy^{m-3} & \pm \&c. \\
 & & \pm ry^{m-3} & \mp \&c.
\end{array}
$$

Præterea quia terminus tollendus debet evanefcere, fiat æqualis zero, reductaque ut moris eft æquatione, valor incognitæ x obtinebitur.

Sic fi terminum fecundum exterminare libeat, fiat

$$- mzy^{m-1} \pm py^{m-1} = 0;\ \text{unde} - mz = \mp p,\ \text{feu}$$

$$z = \pm \frac{p}{m};\ \text{ex quo fluit.}$$

Regula. Si terminus fecundus fit pofitivus, radix augeatur, fi negativus, minuatur quantitate cognita fecundi termini per exponentem primi divifa. Si tertium terminum tollere cupias, fac

$$\frac{m\,.\,\overline{m-1}}{2} z^{2} y^{m-2} \mp (m-1) pzy^{m-2} \pm qy^{m-2} = 0.$$

Sive $z^{2} \mp \frac{2pz}{m} \pm \frac{2q}{m\,.\,m-1} = 0$, unde pro cafibus omnibus quatuor fequentes eliciuntur valores

z^{m}

$$x^m \pm px^{m-1} + qx^{m-2} \&c. \ldots z = \pm \sqrt{\left(\frac{p^2}{mm} - \frac{2q}{m.\overline{m-1}}\right)} \pm \frac{p}{m}.$$

$$x^m \pm px^{m-1} - qx^{m-2} \&c. \ldots z = \pm \sqrt{\left(\frac{p^2}{mm} + \frac{2q}{m.\overline{m-1}}\right)} \pm \frac{p}{m}.$$

Si terminus quartus ſit auferendus; ut incognitæ z valor habeatur, oportet æquationem cubicam reſolvere, quod ex æquationis generalis transformatæ inſpectione ſtatim deprehenditur. Si terminus quintus ſit eliminandus, æquatio quarti gradus occurrit reſolvenda. Generatim pro quocumque termino auferendo, tantæ dimenſionis æquatio eſt reſolvenda, quantam indicat numerus ipſius termini unitate mulctatus.

Exemplum I. Sit æquatio quadratica $x^2 + px - q = 0$; ex qua ſecundus terminus auferri debeat. Quoniam $m = 2$, & ſecundus terminus eſt poſitivus, radices æquationis augeri debent quantitate $\frac{1}{2}p$. Fiat igitur $x + \frac{1}{2}p = y$; erit $x = y - \frac{1}{2}p$; $x^2 = y^2 - py + \frac{1}{4}p^2$; valoribuſque ſuffectis, prodit

y^2

$$\left.\begin{array}{r} y^2-py+\frac{1}{4}p^2 \\ +py-\frac{1}{2}p^2 \\ -\ q \end{array}\right\}=0$$

$$y^2 \ * \ -\frac{1}{4}p^2-q=0$$

$y=\pm\sqrt{\frac{1}{4}p^2+q}$; sed $y=u+\frac{1}{2}p$; ergo

$u=\pm\sqrt{\frac{1}{4}p^2+q}-\frac{1}{2}p$; quod congruit cum iis, quæ de æquationum secundi gradus resolutione docuimus (307.) adeoque etiam per secundi termini evanescentiam habetur methodus æquationes quadraticas cum simplices, tum derivativas resolvendi.

Exemplum II. Sit $u^3-pu^2+qu-r=0$, ex qua pariter secundus terminus exturbandus.

Ob secundum terminum negativum radices æquationis erunt minuendæ quantitate $\frac{1}{3}p$; quare $u-\frac{1}{3}p$ $=y$; $u=y+\frac{1}{3}p$, $u^2=y^2+\frac{2}{3}py+\frac{1}{9}p^2$, $u^3=$ $y^3+py^2+\frac{1}{3}p^2y+\frac{1}{27}p^3$; factaque terminorum substitutione, habetur y^3+

$$\left.\begin{aligned} &y^3 + py^2 + \frac{1}{3}p^2y + \frac{1}{27}p^3 \\ &\quad - py^2 - \frac{2}{3}p^2y - \frac{1}{9}p^3 \\ &\qquad\quad + qy + \frac{1}{3}pq \\ &\qquad\qquad\quad - r \end{aligned}\right\} = 0$$

$$\left.\begin{aligned} &y^3 * - \frac{1}{3}p^2y - \frac{2}{27}p^3 \\ &\qquad + qy + \frac{1}{3}pq \\ &\qquad\qquad - r \end{aligned}\right\} = 0$$

Si igitur $p = 9$, $q = 26$, $r = 34$, æquatio proposita foret $x^3 - 9x^2 + 26x - 34 = 0$; transformata cum secundi termini absentia $y^3 * - y - 10 = 0$.

Exemplum III. Sit $x^3 - 4x^2 + 4x - 6 = 0$ tertio termino mutilanda.

Quia $m = 3$, $p = 4$, $q = 4$, $r = 6$, quantitas, qua radices æquationis datæ sunt augendæ, vel minuendæ, erit $-\frac{1}{3}p \pm \frac{1}{3}\sqrt{p^2 - 3q}$, videlicet $-\frac{4}{3} \pm \frac{2}{3} = -\frac{2}{3}$, vel $= -2$. Fiat igitur $x - \frac{2}{3} = y$;

erit

erit $x = y + \frac{2}{3}$; quare transformata evadet (445.).

$$\left.\begin{array}{r} y^3 + 2y^2 + \frac{4}{3}y + \frac{8}{27} \\ - 4y^2 - \frac{16}{3}y - \frac{16}{9} \\ + 4y + \frac{8}{3} \\ - 6 \end{array}\right\} = 0$$

$$y^3 + 2y^2 \quad * \quad - \frac{130}{27} = 0.$$

Itidem si ponatur $x - 2 = y$, adeoque $x = y + 2$, eodem ordine procedendo, æquatio transformata tertio termino carens invenietur esse $y^3 + 2y^2 * - 6 = 0$.

480. *Scholion.* Si cui volupe fuerit ea persequi, quæ cel. Tschirnhausen pro numero quolibet terminorum intermediorum ex data æquatione auferendo protulit in *Act. Erudit. An.* 1683. *pag.* 204., is ejus methodum generalem in æquationibus superioribus frustra tentabit. Quin etiam in æquationibus ipsis cubicis aliam methodum, quam ipse proponit, vitiosam & imperfectam demonstrat D. Prestet (*nouveaux Elem. des Mathem. Vol.* 2. *Livr. IX. pag.* 413.).

481.

481. Propositio IX. *Æquationem, in qua termini nonnulli deficiunt, in aliam terminis omnibus absolutam transformare.*

Resolutio. Cum terminus aliquis in æquatione deficit, aut radices veræ sunt falsis æquales, aut productum ex radicibus veris in falsas æquatur producto ex veris in veras, vel falsis in falsas, binis, vel ternis, vel quaternis &c. radicibus invicem ductis (414): Ergo si una ex veris, aut una ex falsis augeatur, vel imminuatur, æquationis termini deficientes restituuntur. *Quod erat* &c.

Exemplum. Sit complenda æquatio $x^3 - px^2 * - q = 0$. Statue $x \pm m = y$; erit $x = y \mp m$; & substituendo fit

$$\left.\begin{array}{r} y^3 \mp 3my^2 + 3m^2y \mp m^3 \\ - py^2 \pm 2mpy - m^2p \\ - q \end{array}\right\} = 0.$$

Si æquatio complenda optetur evecta ad altiorem dimensionem, quilibet ejusdem terminus per eam incognitæ potestatem multiplicetur, quæ requiritur, ut expetita emergat dimensio, tum exposita operatio instituatur. Sic si detur æquatio $x^4 - a^4 = 0$ in aliam sexti gradus transformanda, fiat $x^6 - a^4x^2 = 0$, deinde termini deficientes ut supra compleantur.

Ceterum æquationum transformationes magno sunt interdum adjumento pro earum radicibus indagandis, ut duo exempla sequentia demonstrant.

I. Sit $x^3 - 6xx + 11x - 6 = 0$. Pone $x - 2 = y$; erit $x = y + 2$; substitue valores, & transformata evadet $y^3 - y = 0 = y(y+1)(y-1)$, cujus ideo

factores sunt $y = 0, y = -1, y = 1$; quare propositæ radices erunt $0 + 2, -1 + 2, 1 + 2$, seu $2, 1, 3$.

II. Similiter si in æquatione $x^3 - 3x^2 + 3x - 4 = 0$ secundum terminum exturbes, nova æquatio fit $y^3 - 3 = 0$, adeoque $y = \sqrt[3]{3}$, ac proinde $x = 1 + \sqrt[3]{3}$.

482. Lemma. *Data æquatione quavis* (A), *cujus radices sint omnes reales, si ex hac accipiantur tres termini continui, quorum extremi eodem signo afficiantur, quorumque coëfficientes sint* F, G, H, *ex. gr.* $Fx^{m-n} - Gx^{m-n-1} + Hx^{m-n-2}$; *dico, fore* $G^2 > F \times H$.

(C)

(C) $m-1,\ m-2,\ m-3 \ldots\ldots m-n,\ m-n-1,\ m-n-2 \ldots\ldots 1,\ 0$

(A) $x^{m} + ax^{m-1} + bx^{m-2} \ldots\ldots + Fx^{m-n} - Gx^{m-n-1} + Hx^{m-n-2} \ldots\ldots + px^{m-r} + qx^{m-r-1} = 0$

(B) $0,\ 1,\ 2 \ldots\ldots\ldots m+1,\ m+2,\ m+3 \ldots\ldots\ldots m+n,\ m+n+1$

Dem.

Dem. Tam infra quam supra æquationem (A) scribantur, ut in paradigmate, duæ progressiones arithmeticæ (B), (C) ordinem contrarium servantes, ita ut in progressionem inferiori ascendente (B) zero cadat sub primo æquationis termino x^m, & in superiori descendente (C) sub ultimo. Fiat præterea per progressionem inferiorem de more multiplicatio. Propositæ primus terminus evanescet in producto, quo posito $=0$, prodit æquatio secunda, cujus radices erunt omnes reales (470.). Eadem operatio in hac æquatione secunda per progressionem superiorem instituatur, & ultimo termino evanescente, æquatio ad zero per productum hinc emergens facta exhibebit & ipsa radices omnes reales (471.). Finge, continuari hujusmodi operationes, posito semper per vices zero sub terminis extremis æquationum successive nascentium, donec omnes talium æquationum termini evanescant, præter recensitum trinomium, cujus coëfficientes ex dictis multiplicationibus prodeuntes ponamus, esse F, G, H. Certum est, æquationem $Fx^{m-n} - Gx^{m-n-1} + Hx^{m-n-2} = 0$ omnes continere radices reales, quod de quovis alio consimili trinomio, cujus coëfficientes sint P, Q, R, & in quo incognitæ exponentes sint in qualibet progressione arithmetica, semper sit verum. Hisce stantibus, dividatur æquatio $Fx^{m-n} - Gx^{m-n-1} + Hx^{m-n-2} = 0$ per x^{m-n-2}; fiet $Fx^2 - Gx + H = 0$,

$= 0$, seu $x^2 - \frac{G}{F}x + \frac{H}{F} = 0$. Quum autem esse debeat $\frac{G^2}{4F^2} > \frac{H}{F}$, ut radicum realium casus exigit (308.), erit $G^2 > 4F \times H$, ac potiori jure $G^2 > F \times H$, quod de ceteris sic genitis trinomiis semper demonstratur. Ergo &c.

483. *Coroll.* 1. Hinc ex æquatione cujuscumque gradus æquationem non modo quadraticam, sed cubicam &c. statim elicere possumus.

484. *Coroll.* 2. Sumpta igitur quantitate qualibet k, erit $kG^2 > kF \times H$, seu dividendo per GH, erit $\frac{kG}{H} > \frac{kF}{G}$; quare si statuatur $kF > G$, seu $\frac{kF}{G} > 1$, multo magis erit $\frac{kG}{H} > 1$, seu $kG > H$.

485. Propositio X. *In qualibet æquatione ordinata, cujus omnes radices sunt reales, tot erunt radices falsæ, seu negativæ, quot sunt signorum consecutiones, ac tot veræ, seu positivæ, quot sunt signorum permutationes.*

Dem. Si æquatio, de qua sermo est, aliquo termino careat, finge hanc redditam esse completam (481.), hac tamen cautione, ut radices auctæ vel imminutæ persistant ut ante positivæ, aut negativæ (449.), nec refert, an sic agendo fractio aliqua in coëfficientes inrepat, nam de signis, non de coëfficientibus est quæstio. Esto itaque talis æquatio completa

$$x^{m} + Ax^{m-1} + Bx^{m-2} \ldots \&c. + Fx^{m-n} - Gx^{m-n-1} \pm Hx^{m-n-2} \ldots \&c. - Nx^{3} + Px^{2} + Qx + R = 0$$

multiplicanda per $x + k$

fit productum

$$\left.\begin{array}{l} x^{m+1} + Ax^{m} + Bx^{m-1} \ldots \&c. + Fx^{m-n+1} - Gx^{m-n} \pm Hx^{m-n-1} \ldots \&c. - Nx^{4} + Px^{3} + Qx^{2} + Rx + kR \\ + k \ldots + kA \ldots \&c. + \ldots + kF \ldots - kG \ldots \&c. \pm \ldots - kN \ldots + kP \ldots + kQ \ldots \end{array}\right\} = 0$$

Do-

Donec autem in singulis terminis productum istud efformantibus eadem signa permanent, liquet, nullam adesse inter productum & propositam signorum diversitatem. Incipiat jam in producti termino $\frac{-Gx^{m-n}}{+kF}$ signorum variatio. Tres tantum casus hic occurrere possunt; vel enim esse potest $kF = G$, vel $kF < G$, vel $kF > G$. Quoad primum, concipiamus, ita mutari k, ut productum evadat cum iisdem signis completum, vel propositæ æquationis radices ita augeri aut imminui, ut eadem signa recens æquatio retineat. Sic eo deducta res erit, ut duo tantummodo casus, nempe $kF < G$, & $kF > G$ veniant considerandi. Sed si foret $kF < G$, idem signum superius, hoc est $-G$, permaneret, ac proinde nulla usque adhuc fieret signorum mutatio, quæ in termino tantum proxime succedente inciperet. Sit igitur $kF > G$.

Vel permutationem $+ Fx^{m-n} - Gx^{m-n-1}$ excipit in proposita consecutio $-Gx^{m-n-1} - Hx^{m-n-2}$, vel alia permutatio $-Gx^{m-n-1} + Hx^{m-n-2}$. Si primam; fieret ut ante in producto una consecutio, & una permutatio. Si secundum; quoniam etiam $kG > H$ (414.), fieret rursus una consecutio, & una permutatio. Ex quo apparet, quod quicumque sit, & ubicumque repetatur in proposita permutationum continuatarum numerus, una tantum permutatio in producto, post dictam multiplicationem, in conse-

cutionem migrabit. Quare aut istæ permutationes in proposita terminabunt in unam consecutionem, quæ in producto in permutationem (ut ex paradigmatis inspectione constat) transiens, numerum permutationum tam in producto, quam in proposita restituet, & hoc semper; aut istæ permutationes ab earum aliquo exordio usque ad ultimum terminum in proposita, adeoque usque ad producti terminum, qui ultimo propositæ subscribitur, continuabunt, ac quum ultimus producti terminus eodem signo cum ultimo propositæ afficiatur, & ejusdem producti penultimus signum diversum vi num. 484. habere debeat, orietur in producto permutatio, quæ parem permutationum numerum tam in producto, quam in proposita suppeditabit.

Si multiplicatio per $x-k$ instituatur, tot reperientur consecutiones in producto, quot in proposita, quod etiam dignoscitur, si terminorum parium signis in proposita mutatis, multiplicatio per $x+k$ facta ut ante concipiatur, radices enim veræ in falsas, & vicissim commutantur (443.).

Hinc tandem colligitur, permutationum & consecutionum numerum quoad signa idem esse tam in proposita simul cum suo factore, quam in producto. Sed quæcumque proposita æquatio spectari potest tanquam productum binomiorum $x+k$, vel $x-k$ invicem successive multiplicatorum (410.), ex quo ipsa tot acquirat consecutiones, seu tot radices falsas, quot sunt binomia $x+k$, ac tot permutationes, seu tot radices veras, quot sunt binomia $x-k$; ergo &c. *Quod erat* &c.

486.

486. *Coroll.* 1. Quævis æquatio, cujus termini sunt per vices positivi, & negativi, habere potest radices imaginarias, nullas tamen radices reales negativas; & contra quælibet æquatio, cujus omnes termini sint positivi, habere potest radices imaginarias, nunquam tamen radices positivas, quod in primo casu nulla adesse potest consecutio, in secundo nulla permutatio signorum.

487. *Coroll.* 2. Æquatio composita ex quibusvis, & quotcumque æquationibus radices reales habentibus, tot continebit signorum permutationes, & consecutiones, quot radices positivæ & negativæ in æquationibus componentibus insunt; idem enim contingit ac si componentium singuli factores invicem successive multiplicentur.

488. *Coroll.* 3. Si terminus aliquis in proposita deficiens in medio consistat duorum terminorum eodem signo affectorum, ibi radices imaginariæ certissime aderunt, quia statuendo terminum istum primo $=+0$, deinde $=-0$, duo orirentur in proposita signorum status sibi contradicentes, ut rem tentanti palam fiet, quod Regulam demonstratam everteret.

489. *Coroll.* 4. Idem ob eandem rationem est concludendum, si duo, vel plures termini continui deficiant.

490. *Coroll.* 5. Si proposita per binomium quodlibet $x+k$, vel $x-k$ multiplicata, productum hinc emergens eundem non servet consecutionum, ac permutationum numerum, quem exhibet proposita simul cum factore, hoc indicat, propositam radicibus imaginariis laborare.

491.

491. *Coroll. 6.* Si in æquationis propositæ transformata, quæcumque ea sit, terminus aliquis, vel plures intra duos terminos eodem signo affectos disjunctim deficiant; pariterque si duo vel plures termini continui in transformata quomodocumque evanescant, certum erit, propositam radices imaginarias comprehendere. Etenim transformatæ radices sunt radices æquationis propositæ quantitate aliqua reali auctæ, aut imminutæ, vel per quantitatem aliquam realem multiplicatæ, aut divisæ, quod radicum imaginariarum naturam non immutat; adeoque si radices imaginariæ latent in proposita, aderunt quoque in transformata, & vicissim.

CAPUT III.

DE RESOLUTIONE ÆQUATIONUM FACTORES RATIONALES HABENTIUM.

492. PRopositio I. *Æquationis datæ divisores lineares, sive radices simplices, si quæ adsunt, indagare.*

Resolutio. Quoniam ultimus æquationis compositæ terminus est productum ex quantitatibus cognitis omnium æquationum simplicium componentium (372.), si proposita æquatio divisorem habeat linearem, iste constabit ex incognita, addito, demptove aliquo ultimi termini divisore rationali. Resolvendus est igitur ultimus terminus in suos divisores integros rationales (28.), tum per incognitam, additis vel demptis hisce divisoribus, tentanda æquationis re-

resolvendæ divisio, quæ si perfecte succedat, ille ultimi termini divisor erit una ex radicibus quæsitis, quod etiam dignoscitur, si divisore illo in datam æquationem pro incognita introducto, ipsa æquatio evanescat (410. n. 1.). Quod si inter ultimi termini divisores nullus comperiatur, qui additus incognitæ, vel ab ea subductus æquationem datam perfecte dividat, sive loco incognitæ in eandem æquationem introductus eam in nihilum redigat, indicio est, propositam nulla radice rationali præditam esse. *Quod erat* &c.

Exemplum. Sit $x^4 - 11x^3 + 40x^2 - 52x + 16 = 0$. Divisores ultimi termini sunt 1, 2, 4, 8, 16. Sed adverto, omnes æquationis radices ob continuatas signorum permutationes esse veras (485.); ergo æquationes simplices, per quas divisio tentanda est, fient $x-1=0$, $x-2=0$, $x-4=0$, $x-8=0$, $x-16=0$. Ut autem examen evadat expeditius, divisores isti cum signo positivo pro incognita in æquatione sigillatim substituantur, ut innotescat per quos æquatio evanescat, & invenietur, duos tantum 2, & 4 id efficere; quapropter ut æquatio data deprimatur, dividi potest per alterutram æquationem simplicem $x-2=0$, $x-4=0$; sed compendii gratia præstat illam per æquationem $x^2-6x+8=0$ ex duabus hisce simplicibus invicem ductis conflatam dividere, & habebitur quotus $x^2-5x+2=0$. Proposita igitur, quæ est quatuor dimensionum, ad dimensionem duarum, nempe ad $x^2-5x+2=0$, reducta est; cujus divisio est tentanda per divisores ultimi termini æquationis reductæ, videlicet per

$x-1$

$x-1=0$, & per $x-2=0$; sed $x-1$ est prætermittendus, quia si tota æquatio per hunc dividi nequit, idem etiam dictæ ejusdem parti debet accidere; superest igitur $x-2=0$; at quum neque per hanc divisio exacta obtineatur, signum est, reliquas duas radices esse irrationales; profecto per traditas methodos (307.) invenientur esse $\frac{5}{2}\pm\frac{1}{2}\sqrt{17}$. Hæc methodus ad æquationes cujuscumque gradus extendi posset, cum agitur de radicibus rationalibus inveniendis; sed tantis interdum ultimus terminus scatet divisoribus, ut operatio molestissima, & labore improbo plena quemvis ab incepto deterreat; quare ad alias expeditiores methodos est confugiendum.

493. Propositio II. *Radices rationales propius investigare.*

Resolutio I. Si omnes æquationis propositæ radices per signorum consecutiones falsæ, aut si per signorum permutationes veræ deprehendantur, augeantur in primo casu, imminuantur in secundo (445.) quantitate aliqua, quæ ultimum ejus terminum dividat, ita ut in transformata ultimus terminus evadat quantum fieri potest minor ultimo termino æquationis propositæ.

II. Si hujus radices veræ superent falsas, aut contra, satius est eundem in finem prius tentare in primo casu expositam radicum imminutionem, earumque accretionem in secundo.

III. Si radices sint partim veræ, partim falsæ, augeri debent; vel minui quousque ultimus transformatæ terminus quantum fieri potest decrescat. IV.

IV. Ultimus transformatæ terminus sic imminutus resolvatur in suos divisores, iisque auctis vel imminutis quantitate illa, qua propositæ radices imminutæ in primo casu, auctæ in secundo fuerunt, retineantur tantummodo divisores ultimis utriusque æquationis terminis communes.

V. Ex istis reiiciantur illi, qui maximum coëfficientem negativum unitate auctum superant.

VI. Per divisores superstites tentetur in proposita radicum inventio.

VII. Quod si radices æquationis propositæ variis modis augendo vel minuendo, ultimus terminus in transformata semper augeatur, signum est, adesse in proposita radices irrationales.

Exemplum I. Sit $x^4 - 22x^3 + 171x^2 - 542x + 560 = 0$. Quoniam omnes radices sunt veræ, imminuantur quantitate, quæ ultimum terminum 560 dividat sine residuo; ex. gr. fiat $x - 4 = y$, eritque $x = y + 4$; quare peractis substitutionibus, habetur (396.)

$$\left.\begin{array}{r} y^4 + 16y^3 + 96y^2 + 256y + 256 \\ -22y^3 - 264y^2 - 1056y - 1408 \\ +171y^2 + 1368y + 2736 \\ -542y - 2168 \\ +560 \end{array}\right\} = 0$$

Summa $y^4 - 6y^3 + 3y^2 + 26y - 24 = 0$.
Hæc æquatio transformata habet ultimum terminum

num $24 < 560$, qui eſt ultimus primæ æquationis terminus; ultimi autem termini 24 diviſores (rejectis majoribus quam $6+1$) ſunt 1, 2, 3, 4, 6; ſed omnes iſti diviſores ſunt augendi numero 4; quare evadent 5, 6, 7, 8, 10. Ex his numerus 6 non eſt diviſor ultimi termini 560 primæ æquationis; ergo hoc omiſſo, ſuperſunt 5, 7, 8, 10, per quos evaneſcentia ejuſdem propoſitæ tentari debet; at hoc efficitur per tres tantum numeros 5, 7, 8, per quos inſtituta de more diviſione, pro ultimo quoto prodit $x-2=0$; proinde 2, 5, 7, 8 ſunt radices propoſitæ, cujus per numeros ſolum poſitivos tentavimus evaneſcentiam, quia ipſius radices ſunt omnes poſitivæ.

Exemplum II. Sit $x^4+4x^3-19x^2-106x-120=0$. Quoniam ex ſignis tres ſunt radices falſæ, & una vera (485.), iſtæ augeantur quantitate, quæ ultimum terminum perfecte dividat, ex. gr. numero 3. Fiat igitur $x+3=y$; erit $x=y-3$; hinc (445.)

$$\left.\begin{array}{r} y^4-12y^3+54y^2-108y+81 \\ +4y^3-36y^2+108y-108 \\ -19y^2+114y-171 \\ -106y+318 \\ -120 \end{array}\right\}=0$$

$$y^4-8y^3-y^2+8y\ *=0.$$

Quare dividendo per y, fiet $y^3-8y^2-y+8=0$. Una igitur ex propoſitæ radicibus eſt -3. Jam vero ul-

ultimi termini transformatæ divisores sunt 1, —1, 2, —2, 4, —4, 8, —8, quos cum affirmative, tum negative accepimus, quod ex signis radices sunt partim veræ, partim falsæ. Divisores isti numero 3 imminuti evadunt —2, —4, —1, —5, 1, —7, 5, —11. Ex his sunt reiiciendi —7, & —11, quippe qui ultimum æquationis propositæ terminum accurate non dividunt; quapropter si reliqui in propositam loco *x* sigillatim introducantur, tres tantum, nimirum —2, —4, +5 illam in nihilum redigent; ac proinde propositæ radices erunt —2, —3, —4, +5.

494. *Scholion.* Quoniam ultimus transformatæ terminus (suffecta incognita pro radice) cum proposita congruit (445. n. 1.) si (1.°) scire statim cupias, an quantitas, per quam æquationis propositæ radices augentur, aut imminuuntur, efficiant in transformata ultimum terminum minorem ultimo propositæ termino; substitue in proposita quantitatem illam in locum incognitæ, & per id quod provenit, voti fies compos. Sic in primi Exempli proposita substituendo pro *x* quantitatem 4, habebis — 24 pro ultimo transformatæ termino. (2.°) Si proposita per hujusmodi substitutionem evanescit, scimus, quantitatem suffectam unam esse ex radicibus propositæ (410. n. 1.), quæ idcirco una dimensione deprimetur (410. n. 2.). (3.°) Si in transformata evadunt radices nonnullæ negativæ, quæ in proposita erunt positivæ, aut vicissim; tunc in primo casu quantitas adsumpta, qua radices in proposita fuerant imminutæ, erit major totidem radicibus positivis, & contra (447. 448.). Sic quoniam

niam in propoſita primi Exempli omnes radices erant poſitivæ, & in transformata una occurrit negativa, patet aliquam ex dictis poſitivis eſſe minorem quam 4, quæ eſt quantitas, qua ipſæ radices fuerunt imminutæ. Quod ſi radices imminutæ fuiſſent quantitate 8, ultimus terminus in transformata evanuiſſet, ac transformata ſuperſtes reliquas radices omnes negativas contineret, adeoque una propoſitæ radix foret 8, & reliquæ minores quam 8, qui proinde numerus limitem non excedendum in earum perquiſitione conſtitueret. Verum alia etiam adeſt methodus præclara, quæ arctiores limites exhibet, ad quam exponendam progredimur.

495. Propoſitio III. *Æquationum radices, vel factores ſimplices alio modo indagare.*

Reſolutio. Eſto n cum ſuo ſigno unus ex diviſoribus ultimi termini æquationis propoſitæ. Eſto $x+n=0$ unus ex factoribus ejuſdem æquationis. Si pro x numerus quilibet m ſubrogetur, certum eſt, æquationem diviſibilem fore per $m+n$. Sit enim $x^2+x-6=0$; quum 2 ſit diviſor ultimi termini 6, & æquationis factor $x-2=0$, patet, poſito m pro x, æquationem quoque $m^2+m-6=0$ fore diviſibilem per $m-2$. Hoc ſtante, detur æquatio generalis $x^c+px^{c-1}+qx^{c-2}+rx^{c-3}\ldots+T=0$, in qua poſito n uno ex diviſoribus ultimi termini T, ſit $x+n$ unus ex ejuſdem æquationis factoribus. Sumptis pro x valoribus 3, 2, 1, 0, −1, &c. progreſſionem arithmeticam ſervantibus, iſti pro x tam in

æqua-

æquatione generali, quam in ſactore $x+n=0$ ſucceſſive ſubſtituantur, prout ſequens monſtrat Tabella

A Hypotheſes	*B* Subſtitutiones	C Diviſores
$x=3$	$3^{c}+p3^{c-1}+q3^{c-2}+r3^{c-3}\ldots\ldots+t$	$3+n$
$x=2$	$2^{c}+p2^{c-1}+q2^{c-2}+r2^{c-3}\ldots\ldots+t$	$2+n$
$x=1$	$1+p+q+r\ldots\ldots\ldots\ldots+t$	$1+n$
$x=0$	$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+t$	$0+n$
$x=-1$	$\pm 1\mp p\pm q\mp r\ldots\ldots\ldots\ldots+t$	$-1+n$

In prima columna *A* recenſitæ hypotheſes apparent. In ſecunda *B* hypotheſium pro æquationis generalis incognita ſubſtitutiones. In tertia *C* diviſores dictis ſubſtitutionibus e regione reſpondentes; ubi animadverte, diviſores quoque iſtos procedere in progreſſione arithmetica, cujus differentia eſt itidem unitas. Ergo quum hypotheſis $x=0$ det tantummodo n, qui eſt unus ex diviſoribus ultimi termini æquationis propoſitæ; ſi æquationis hujus incognita in valores $1, 0, -1$ ſucceſſive commutetur, ita ut hypotheſis prima ſit $x=1$, ſecunda $x=0$, tertia $x=-1$, nullus ex ultimi termini æquationis hujuſce diviſoribus, quos ſufficit hypotheſis $x=0$, eſſe poterit radix quæſita, niſi ſit unitate minor quam aliquis ex iis, quos præbet hypotheſis $x+1$, vel unitate major quam aliquis ex iis, quos exhibet hypotheſis $x=-1$. Ex quo fluit *Regula*. Pro datæ æquationis incognita ſufficiantur termini progreſſio-

nis arithmeticæ 1, 0, —1. Singulorum hinc emergentium aggregatorum scribantur ex ordine omnes divisores, qui tam negativi, quam positivi sunt pro re nata considerandi. Hos inter, ceteris rejectis, illi seligantur in progressione arithmetica procedentes, quorum differentia communis est unitas. Termini hujusce progressionis facto ex $x = 0$ prodeunti respondentes erunt valores incognitæ x addendi ad exactam æquationis datæ divisionem tentandam. *Quod erat* &c.

496. *Coroll.* 1. Si recensitæ progressionis arithmeticæ (in quam abeunt divisores columna *C* contenti) numeri crescant, valores n sumantur affirmativi, & contra negativi si decrescant; quum enim hypothesium series sit decrescens, decrescens pariter esse debet series divisorum; ut itaque series crescens fiat decrescens, signa in ejus terminis mutari debent; adeoque in primo casu fit $x = +n$, in altero, $x = -n$, seu in primo casu $x - n = 0$, in altero $x + n = 0$.

497. *Coroll.* 2. Si in hypothesi $x = 0$ duo vel plures progressiones emergant ad examen revocandæ, ponatur, ad ambiguitatem tollendam, $x = 2$, vel $x = -2$, & notentur divisores, qui progressiones antecedentes continuant, iique retineantur, ceteris hac conditione carentibus prætermissis; tum numerus ille in finem præfixum tantummodo adhibeatur hypothesi $x = 0$ respondens, qui plures seriei terminos ingreditur. Sin autem invitis hisce hypothesibus, ob æquales progressionum terminos numerus neuter excludi potest, ponatur $x = 3$, aut

$x =$

$x = -3$, & reliqua ut ante peragantur, & si deinceps, donec progressio magis continuata minus continuatam excludat.

498. *Coroll.* 3. Cum maximi æquationis termini coëfficiens non est unitas, nec ipsum ad unitatem reducere libeat, tunc hujusmodi coëfficientis divisore posito $= m$, divisores in columna C collocandi evadent $3m + n$, $2m + n$, $m + n$, $0 + n$, $-m + n$, adeoque semper servabitur progressio arithmetica, cujus differentia communis erit m, divisor nempe coëfficientis maximi termini æquationis datæ, qui proinde, statuto prius numero n, facile per divisorum proximorum subtractionem, ut evidens est, reperitur; hoc autem per x multiplicato, si huic producto addatur numerus pro n selectus, fiet binomium, per quod æquationis datæ tentanda est divisio.

499. *Monita.* (1.) Si æquatio proposita ultimum terminum nimis habeat compositum, ejus radices augeri prius, vel minui possunt (493.) ad ultimum terminum in transformata imminuendum; tum exposita operatio instituenda, ut radices æquationis propositæ ex transformatæ radicibus facilius eruantur.

(2.) Quotiescumque data æquatio per aliquam hypothesim evadit $= 0$, notum est, in hac hypothesi æquationis radicem contineri (410. n. 1.).

(3.) Quum zero omnes numeros pro suis divisoribus agnoscat (quicumque enim numerus dividens zero dat semper zero pro quoto), nullum numerum in dictis progressionibus texendis potest excludere.

Hæc autem omnia per exempla subsequentia clarius percipientur.

Exemplum I. Sit æquatio $x^3-2x^2-13x+6=0$.

A	*B*	*C*	*D*	
Hypoth.	*Substit.*	*Divisores*	*Progressiones*	
$x=2$	20		1	5
$x=1$	8	1, 2, 4, 8	2	4
$x=0$	6	1, 2, 3, 6	3	3
$x=-1$	16	1, 2, 4, 8, 16		2
$x=-2$	16			1

Colloco in prima columna *A* hypotheses, substituendo primum 1, deinde zero, tum -1 pro incognita x in æquatione proposita, & obtineo primum 8, deinde 6, tum 16, quos numeros, signis prætermissis, scribo in columna *B*. Horum divisores omnes quæro (28.), quos pono in columna C. Animadverto, utrum inter hypothesis $x=0$ divisores aliquis comperiatur, cui si addatur unitas, exurgat numerus aliquis degens inter divisores hypothesis $x=1$, cuique si unitas detrahatur, fiat numerus aliquis occurrens inter divisores hypothesis $x=-1$, & viciffim, ita ut formetur progressio arithmetica vel crescens vel decrescens; ac duas hujusmodi progressiones invenio, quas in columna *D* transfero. Quoniam vero qua de duabus hisce progressionibus uti prius debeam nescio, pono (497.) $x=2$; at adhuc quam ex novis progressionibus præferam igno-

ignoro; itaque statuo $x = -2$, ac statim video unam tantum remanere progressionem decrescentem, quare per numerum -3 hypothesi $x = 0$ respondentem tento æquationis evanescentiam; quod quum ex voto cedat, per $x + 3$ divisionem institituo, qua peracta, prodit $x^2 - 5x + 2$ pro quoto; ac proinde in duas æquatio data resolvitur, nempe $x + 3 = 0$, $x^2 - 5x + 2 = 0$, adeoque tres habet radices, nimirum -3; $\frac{5}{2} + \frac{1}{2}\sqrt{17}$; $\frac{5}{2} - \frac{1}{2}\sqrt{17}$.

Exemplum II. Sit $6x^3 - 53x^2 - 24x + 135 = 0$. Absolutis ad Regulæ normam operationibus, obtineo

A	B	C	D				
Hypoth.	*Subſtit.*	*Diviſores*	*Progreſſiones*				
$x = 2$	77	7, 11, 1, 77	1		7	−1	11
$x = 1$	64	1, 2, 4, 8, 16, 32, 64	2	4	8	1	8
$x = 0$	135	1, 3, 5, 9, 15, 27, 45, 135	3	3	9	3	5
$x = -1$	100	1, 2, 4, 5, 10, 20, 25, 50, 100	4	2	10	5	2
$x = -2$	77			1	11	7	−1

Incertus igitur quinam sint numeri ab hypothesibus $x = 1$, $x = 0$, $x = -1$ dimanantes eligendi, addo hypotheses $x = 2$, & $x = -2$, ac tres mihi obveniunt progressiones, quæ magis continuatæ reliquas excludunt. Sumpto itaque primæ crescentis 7, 8, 9, 10, 11 nu-

numero 9 hypothesi $x = 0$ respondente, per $x - 9$ æquationis datæ divisionem tento (quia quum progressionis differentia communis sit 1, incognita x in unitatem ducta eadem remanet), & exurgit quotus $6x^2 + x - 15$. Sed rem exploro etiam per secundæ progressionis pariter crescentis numerum 3; est autem in hac progressione differentia communis $= 2$; ergo per $2x - 3$ æquationem divido, & fit quotus $3x^2 - 22x - 45$. Curiositati postmodum indulgens idem exequor tertiam excipiendo progressionem decrescentem, cujus differentia communis 3, adeoque per $3x + 5$ divisionem ut ante instituo, & habeo pro quoto $2x^2 - 21x + 27$. Ergo æquationis datæ radices sunt $9, 3, -5$, factores $x - 9, 2x - 3, 3x + 5 = 0$.

Verum fieri potest, ut æquatio data nullo factore simplici gaudeat, quo in casu allata methodus irrita evadit. Tentandum est igitur, an divisores quadratici per methodum consimilem haberi possint; pro quo sit.

500. Propositio IV. *Æquationis propositæ factores quadraticos rationales, si qui adsunt, investigare.*

Resolutio. Statue, $mx^2 + nx + f$ esse æquationis propositæ factorem quadraticum. Exaratis de more in columna *A* pro x hypothesibus, prout in apposita tabella videre est, exprimat columna *B*, ut supra, substitutiones talium hypothesium in proposita, & columna *C* alias hujusmodi substitutiones in dicto factore peractas repræsentet.

A

A	*B*	C	*D*	*E*
Hypoth.	*Subſtit. prima*	*Subſtit. ſec.*	*Hypoth. quadr.*	*Differentiæ*
$x = 3$		$9m + 3n + f$	$9m$	$3n + f$
$x = 2$		$4m + 2n + f$	$4m$	$2n + f$
$x = 1$		$m + n + f$	m	$n + f$
$x = 0$		$\ldots\ldots + f$	0	$0 + f$
$x = -1$		$m - n + f$	m	$-n + f$
$x = -2$		$4m - 2n + f$	$4m$	$-2n + f$

Tum collocentur e regione in columna *D* quadrata numerorum in hypotheſibus aſſumptorum ducta in m, qui aliquem ex diviſoribus coëfficientis primi termini deſignat. Hiſce poſitis, animadverto, quod ſi hujuſmodi quadrata ex ſecundis ſubſtitutionibus in columna *C* notatis auferantur, reſidua, quæ in columna *E* apparent, progreſſionem arithmeticam formant, cujus differentia communis eſt n; quapropter ſi æquatio data factorem habeat ſecundi gradus, quoniam m ſemper innoteſcere poteſt, ſequens ex præmiſſis profluit.

Regula. (1.°) In æquatione propoſita termini progreſſionis arithmeticæ $1, 0, -1$ pro incognita ſubſtituantur; (2.°) diviſores omnes ex harum hypotheſium factis emergentes inveniantur; (3.°) quadrata terminorum $1, 0, -1$ per diviſorem aliquem m coëf-

ficientis altiſſimo termino propoſitæ præfixi multiplicata a diviſoribus jam inventis, tam affirmative, quam negative ſumptis, ſubtrahantur, & differentiæ ex ordine diſponantur; (4.°) progreſſiones omnes arithmeticæ ab hiſce reſiduis ſuppeditatæ accipiantur; (5.°) terminus harum progreſſionum hypotheſi $x = 0$ reſpondentium cum ſuo ſigno ſtabit pro f; progreſſionis ejuſdem differentia prodiens ex ſubtractione alicujus termini ab antecedente cum ſuo ſigno ſtabit pro n; diviſor coëfficientis, quo altiſſimus æquationis terminus afficitur, ſtabit pro m; quibus

A	B	C	D	
Hypoth.	*Subſtit.*	*Diviſores*	*Quadr. Hyp.*	
$x = 3$	12	1, 2, 3, 4, 6, 12	9	3, –3, –5, –6, –
$x = 2$	12	1, 2, 3, 4, 6, 12	4	8, 2, 0, –1, –2,
$x = 1$	8	1, 2, 4, 8	1	7, 3, 1, 0, –2, –
$x = 0$	6	1, 2, 3, 6	0	6, 3, 2, 1, –1, –
$x = -1$	12	1, 2, 3, 4, 6, 12	1	11, 5, 3, 2, 1, 0, –

Scriptis hypotheſibus in columna A, & omnibus diviſoribus ſubſtitutionum in columna B extantium in columna C diſpoſitis, numerorum 2, 1, 0, –1 quadrata in columna D ſita ab illis tam affirmative, quam negative ſumptis ſubduco, & reſidua in columna E colloco, ex quo ſex obtineo progreſſiones in columna F notatas, quarum ut inutiles exclu-

omnibus in formula mx^2+nx+f suffectis, exurget factor, per quem propositæ divisio tentanda; (6.°) quod si plures adsint progressiones arithmeticæ, quarum superfluas excludere velis, pone successive $x=2$, vel $x=-2$, vel $x=3$, vel $x=-3$, &c. ac progressiones magis continuatas eligendo, utiles ab inutilibus secernes. *Quod erat* &c.

Exemplum I. Sit $x^4-5x^3+7x^2-5x-6=0$.

Quia nullus in hac æquatione factor simplex reperitur, operatio ad Regulæ novissimæ formam est sequens

E	F					
Differentiæ	*Progressiones*					
, −10, −11, −12, −13, −15, −21	−11	−3				
; , −6, −7, −8, −10, −16	−8	−1	0	−1	−1	−3
−9	−5	1	1	−2	0	−2
−6	−2	3	2	−3	1	−1
−4, −5, −7, −13	1	5	3	−4	2	0

dam, adiicio hypothesim $x=3$, debitisque institutis operationibus, duos tantum numeros −11, −3 invenio primam ac secundam progressionem continuantes; quare his progressionibus retentis, ceteras respuo. Quum autem in prima progressione hypothesi $x=0$ respondeat −2, numerum hunc cum suo signo in formula mx^2+nx+f pro f sufficio.

At

At subtracto -2 ex -5, fit -3; ergo -3 pro n in ipsam formulam intromitto. Quoniam vero maximi termini x^4 coëfficiens est unitas, cujus divisor est eadem unitas, loco divisoris m pono 1, ex quo exurgit $x^2 - 3x - 2$. Per hunc factorem divisionem tento, quæ optime succedit, ac pro quotiente dat $x^2 - 2x + 3$, qui etiam prodiisset; si secundam progressionem adhibuissem, tunc enim fuisset $f = 3$, $n = 1 - 3 = -2$, $m = 1$, ut apertum est. Extractis igitur radicibus ab hisce duabus postremis æquationibus quadraticis, propositæ radices invenientur esse $3 \pm \sqrt{17}$, & $1 \pm \sqrt{-2}$.

Exemplum II. Sit $4x^5 + 16x^4 - 22x^3 - 14x^2 - 56x + 77 = 0$.

A	*B*	C	*D*	E	*F*
2	117	1, 3, 9, 13, 39, 117	8	−125, −47, −21, −17, −11, −9, −7, −5, +1, +5, +31, +109	5
1	5	1, 5	2	−7, −3, −1, +3	−3
0	77	1, 7, 11, 77	0	−77, −11, −7, −1, +1, +7, +11, +77	−11
−1	153	1, 3, 9, 17, 51, 153	2	−155, −53, −19, −11, −5, −3, −1, +1, +7, +15, +49, +151	−19
−2	437	1, 19, 23, 437	8	−445, −31, −27, −9, −7, +11, +15, +429	−27

Præ.

Præviis hypothesibus in columna *A* exaratis, numeros, in quos æquatio data ex hypothesibus introductis convertitur, scribo in columna *B*. Hujusmodi numerorum divisores omnes exhibet columna *C*. Primi termini $4x^5$ coëfficiens 4 tres habet divisores 1, 2, 4. Primo 1, utpote inutili, quod nihil immutat, prætermisso, sumo 2, per quem multiplico hypothesium quadrata, & producta noto in columna *D*. Residua, quæ fiunt, numeros columnæ *D* ex numeris columnæ *C* cum affirmative, tum negative sumptis subtrahendo, expono in columna *E*. Examino progressiones arithmeticas postremis hisce numeris inclusas, & unicam invenio, quæ prostat in *F*; quare pro *f* numerum -11 hypothesi $x=0$ respondentem substituo. Seriei arithmeticæ inventæ differentia, dempto -11 ex -3, est 8, quæ stat pro *n*; ergo valoribus in trinomio mx^2+nx+f sufficitis, prodit $2x^2+8x-11$; per hoc divisionem tento, & obtineo quotientem $2x^3-7$. Duæ igitur æquationes propositam mutuo ductu conflantes sunt $2x^2+8x-11=0$, $2x^3-7=0$.

501. *Corollarium.* Quotiescumque igitur æquatio proposita radicem recipit hujus formæ $a+b\sqrt{\pm p}$, ubi a, b, p sunt rationales, quoniam includere quoque debet aliam $a-b\sqrt{\pm p}$ (423.), elicietur semper ab ea factor quadraticus rationalis, nimirum $x^2-2ax+a^2\mp p$.

502. Propositio V. *Æquationis propositæ factores cubicos, si qui adsunt, indagare.*

Re-

Resolutio. Expedit primum de seriebus arithmeticis nonnulla præmonere. *Seriem arithmeticam primi ordinis* dicimus illam, in qua, ut vidimus, terminorum differentiæ sunt constantes, ut $2n+f$; $n+f$; f; $-n+f$; &c. *Seriem arithmeticam secundi ordinis* illam vocamus, in qua terminorum differentiæ seriem arithmeticam primi ordinis constituunt, veluti $4n+2f+g$; $n+f+g$; g; $n-f+g$, &c. cujus termini pro differentiis habent seriei arithmeticæ terminos $3n+f$; $n+f$; $-n+f$, &c. Ex quo conjicitur, quæ sit *series arithmetica tertii ordinis*, quæ *quarti*, &c. Hisce stantibus, factor cubicus quæsitus sit mx^3+nx^2+fx+g.

$x=2$	$8m+4n+2f+g$	$8m$	$4n+2f+g$	
$x=1$	$m+n+f+g$	m	$n+f+g$	$3n+f$
$x=0$	$+g$	0	$+g$	$n+f$
$x=-1$	$-m+n-f+g$	$-m$	$n-f+g$	$-n+f$

Hypothesibus ut ante in prima columna notatis, scribantur in secunda numeri, in quos æquatio data per harum hypothesium substitutiones convertitur. In columna tertia prostent hujusmodi numerorum divisores. Columna quarta (quæ hic est secunda) factores cubicos cum hypothesibus loco x substitutis exponat. In columna quinta (quæ hic est tertia) collocentur cubi multiplicati per m exprimentem unum ex divisoribus coëfficientis primi termini

mini æquationis datæ. Columna sexta (quæ hic est quarta) hujusmodi cuborum a dictis factoribus cubicis respondentibus differentias exhibeat. Columna septima (quæ hic est quinta) differentias secundas comprehendat. His ita dispositis, post inventas secundas differentias, quod præcipuam attentionem requirit, fiat g (hypothesi $= 0$ respondens) $= n + f$, deinde $n+f+g = 3n+f$, tum $n-f+g = -n+f$; his autem valoribus inter se collatis, numerorum n, f, g valores determinari poterunt, quibus in factore cubico suffectis, tenetur æquationis datæ divisio; *Quod erat* &c. Sed ista sequens clarius exponet.

Exemplum. Sit $x^6 - 2x^5 - 9x^4 + 20x^3 - 16x^2 + 3x + 3 = 0$.

$x = 2$

$x = 2$	39	1, 3, 13, −1, −3, −13	8	−7, −5, 5, −9, −11, −21
$x = 1$	0	quilibet numerus	1	quilibet numerus
$x = 0$	3	1, 3, −1, −3	0	1, 3, −1, −3
$x = -1$	42	1, 2, 3, 6, 7, 14, 21, 42, −1, −2, −3, −6, −7, −14, −21, −42	−1	2, 3, 4, 7, 8, 15, 22, 43, 0, −1, −2, −5, −6, −13, −20, −41

Lo-

Locatis in prima Tabellæ columna hypothesibus; in secunda substitutionibus; in tertia divisoribus; in quarta hypothesium cubis, quod primi termini æquationis coëfficiens est unitas; quinta columna exhibeat hujusmodi cuborum subtractiones a respondentibus tertiæ columnæ divisoribus. Hos inter quintæ columnæ numeros duas invenio series arithmeticas secundi ordinis, nimirum

$-11, -1, 1, -5$

$5, \quad 2, 3, \quad 8$

Ut e prima factorem quæsitum eruam pono $n+f+g = -1$; $g=1$; $n-f+g=-5$. Hinc ob $f=-2-n$, nec non $f=6+n$, habetur $n=-4$; $f=2$; quibus valoribus in factore x^3+nx^2+fx+g subrogatis, prodit x^3-4x^2+2x+1, per quem si divisionem tento, res ex voto cedit. Altera series exhibet $n+f+g=2$; $g=3$; $n-f+g=8$, adeoque $n=2$, & $f=-3$; unde emergit factor $x^3+2x^2-3x+3=0$, qui pariter est æquationis propositæ divisor exactus; quapropter ex duobus hisce cubicis factoribus proposita coalescit.

503. *Scholion.* (1.°) Ad indagandum factorem biquadraticum series arithmeticæ tertii ordinis requiruntur, & sic deinceps; quare semper crescit methodi difficultas. (2.°) Præstat ad calculi compendium æquationes propositas ita prius, si oporteat, tractare, ut maxima incognitæ potestas unitatem habeat pro divisore (458.). Si (3.°) terminus aliquis in proposita deficiat, æquatio compleatur (481.), secus accidere potest, ut factor etiam quæsitus eodem termino careat, adeoque methodi allatæ irritæ evadant.

dant. Sic si æquationis $x^5 + 2x^3 - 3x^2 - 35x + 15 = 0$ quæras factorem quadraticum, qui est $x^2 - 5$, nunquam illum per traditam methodum (500.) adsequi poteris.

504. Propositio VI. *Facti cujuslibet incognitam involventis factores indagare.*

Resolutio. Hujusmodi factum esto $x^m + px^{m-1} + qx^{m-2} +$ &c. Ut ejus factores investigentur, pone factum illud $= 0$; inventisque, si fieri potest, æquationis hujus radicibus, habebis quod quæris. Etenim finge, æquationis hujus radices esse a, b, c, &c.; facti propositi factores erunt $x-a, x-b, x-c$, &c. *Quod erat* &c.

CAPUT IV.

DE ÆQUATIONIBUS PLURES RADICES INVICEM ÆQUALES HABENTIBUS.

505. PRopositio I. *Quot adsunt in æquatione proposita radices invicem æquales etiam cum inæqualibus commixtæ, tot in transformata termini, incipiendo ab ultimo, evanescunt.*

Dem. Æquationis datæ factores sint $x - a, x - b, x - c, x + d, x + e$, &c. quare si fiat $x - a = y, x - b = y, x - c = y, x + d = y, x + e = y$, &c. erit $x = y + a, x = y + b, x = y + c, x = y - d, x = y - e$, &c. adeoque transformatæ factores erunt $y + a - x, y + b - x, y + c - x, y - d - x, y - e - x$, &c.

Statue $x = a = b = c$; transformatæ factores evadent $y + a - a, y + a - a, y + a - a, y - d - x, y - e - x$, &c.; ergo quum tres transformatæ factores sint $y + 0$, patet transformatam per totidem dimensiones deprimi posse (419.); ergo tres in ipsa termini evanescent. Pari ratione si plures fuerint in proposita radices invicem æquales, totidem termini in transformata nihilo æquales fient, radicibus inæqualibus nihil obstantibus. *Quod erat* &c.

Aliter. Si in æquatione $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} + \&c. = 0$ pro x nova incognita $y + t$ sufficiatur, habebitur transformata, quæ æquivalebit producto, vel æquationi $(y + t - a)(y + t - b)(y + t - c)(y + t - d) = 0$, cujus radices, seu novæ incognitæ y valores erunt $a - t, b - t, c - t, d - t$, &c. quare si proposita æquatio habeat numerum m radicum æqualium quantitati positivæ, aut negativæ a, ut sit $t = x = a$, etiam transformata eundem habebit numerum m radicum æqualium $a - t = 0$. Hinc ultimus terminus multiplum quantitatis $(t - a)^m = 0$, penultimi termini coëfficiens multiplum quantitatis $(t - a)^{m-1} = 0$, antepenultimi coëfficiens multiplum quantitatis $(t - a)^{m-2} = 0$, & sic deinceps, comprehendet (421.), donec exponens quantitatis $(t - a)$ evadat unitas, quia tunc completur numerus radicum æqualium, & ultimorum terminorum m;

hinc

hinc tot termini in transformata evanescent, quot sunt radices æquales in proposita. *Quod erat* &c.

506. *Coroll.* 1. Ergo si proposita quinque habeat radices æquales $= \pm a$, quatuor $= \pm b$, tres $= \pm c$, duas $= \pm d$; quinque termini comparate ad radices $\pm a$, quatuor comparate ad $\pm b$, tres comparate ad $\pm c$, duo comparate ad $\pm d$ in transformata evanescent. Terminus autem ultimus transformatæ continebit $\pm 5a$, $\pm 4b$, $\pm 3c$, $\pm 2d$; penultimus $\pm 4a$, $\pm 3b$, $\pm 2c$, $\pm d$; antepenultimus $\pm 3a$, $\pm 2b$, $\pm c$, & sic deinceps.

507. *Coroll.* 2. Quum igitur transformata deprimi possit pro maximo radicum invicem æqualium numero; qui adest in proposita, eligi in illa potest terminus evanescens ad minimam dimensionem ascendens pro transformatæ, ac proinde propositæ dictis radicibus inveniendis.

508. *Coroll.* 3. Si æquatio dimensionis n omnes habeat radices ejusdem generis, vel plures diversi generis invicem æquales sine inæqualium mixtione, numerus terminorum in transformata evanescentium erit $n-1$, adeoque sola incognita ad dimensionem n assurgens nihilo æqualis superetit.

509. *Coroll.* 4. Similiter si æquatio radices æquales unius tantum generis contineat, atque ita ipsam transformare libeat, ut absit secundus terminus; tota transformata, excepta incognita ad priorem potestatem elata, evanescet. Hujusmodi enim æquationes sunt totidem potestates (420.); in quibus secundi termini coëfficiens per maximum incognitæ exponentem divisus unam ex æqualibus ejusdem æquatio-

nis radicibus suppeditat. Ut id clarius innotescat, esto $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} +$ &c. Ut hujus æquationis transformata secundo termino careat, fieri debet $x + \frac{p}{n} = y$ (479.); sed p est omnium radicum aggregatum; ergo $\frac{p}{n}$ est una radix ob omnes radices invicem æquales; ergo ponendo $x = y - \frac{p}{n}$, omnes transformatæ termini debent præter primum per demonstrata evanescere. Sic si fuerit $x^3 - 9x^2 + 27x - 27 = 0$, statuendo $x - \frac{9}{3}$, seu $x - 3 = y$, erit $x = y + 3$; adeoque transformata evadet $y^3 = 0$.

510. *Coroll.* 5. Scire igitur facile possumus, an æquatio data omnes habeat radices invicem æquales, & quæ sint.

Exemplum. Esto $x^4 - x^3 - 3x^2 + 5x - 2 = 0$, quæ tres habet radices æquales; istæ ex signis erunt positivæ (485.); quia nullas continere potest æquatio radices imaginarias (424.); fiat igitur $x - r = y$; erit $x = y + r$; quare transformata erit (445.)

$$y^4 \left.\begin{matrix} +4r \\ -1 \end{matrix}\right\} y^3 \left.\begin{matrix} +6r^2 \\ -3r \\ -3 \end{matrix}\right\} y^2 \left.\begin{matrix} +4r^3 \\ -3r^2 \\ -6r \\ +5 \end{matrix}\right\} y \left.\begin{matrix} -r^4 \\ -r^3 \\ -3r^2 \\ +5r \\ -2 \end{matrix}\right\} = 0$$

Quo-

Quoniam hujus tres ultimi termini evanescunt, sumo (507.) $6t^2 - 3t - 3 = 0$, cujus radices sunt 1, & $-\frac{1}{2}$. Sed nullas proposita continet radices fractas (435.), & radices æquales sunt, ut diximus, positivæ; ergo radice $-\frac{1}{2}$ exclusa, superest 1 pro una ex positis radicibus æqualibus in proposita existentibus (506.), quæ idcirco erit divisibilis per $(x-1)$ $(x-1)(x-1) = 0 = x^3 - 3x^2 + 3x - 1$, ut periculum facienti constabit, fit enim quotus $x + 2 = 0$, seu $x = -2$. Quum autem penultimi termini transformatæ coëfficiens continere debeat duas ex dictis radicibus æqualibus [*l. cit.*] erit & ipsa divisibilis per $(t-1)(t-1) = 0 = t^2 - 2t + 1$; & quidem pro quoto prodit $4t + 5 = 0$. Tribus terminis in transformata evanescentibus restat $\left.\begin{matrix} y^4 + 4t \\ -1 \end{matrix}\right\} y^3 = 0$, seu $y + 4t - 1 = 0$, seu $y = 1 - 4t$; sed $t = 1$; ergo $y = -3$, adeoque $x = y + 1 = -2$ ut supra.

511. Propositio II. *Si æquatio proposita plures continens radices invicem æquales ejusdem generis positivas, aut negativas, etiam cum inæqualibus mixtas, per progressionem quamlibet arithmeticam multiplicetur, non solum remanebit nibilo æqualis, sed una dimensione deprimetur.*

Dem. Sit $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} + \&c. = 0$, quæ habeat numerum m radicum æqualium

quantitati r positivæ aut negativæ. Ergo æquatio ista multiplum quantitatis $(x-r)^m$ complectitur (421.). Fiat $x-r=y$; erit $x=y+r$. Itaque si æquatio transformetur, penultimi transformatæ termini cœfficiens, nempe $nx^{n-1}+(n-1)px^{n-2}+(n-2)qx^{n-3}+(n-3)rx^{n-4}+\&c.=0$ (446.), multiplum quantitatis $(x-r)^{m-1}$ comprehendet (505.). Multiplicetur iste coëfficiens per bx; prodibit $nbx^n+(n-1)bpx^{n-1}+(n-2)bqx^{n-2}+(n-3)brx^{n-3}+\&c.=0$. Multiplicetur præterea proposita per a; habebitur $ax^n+apx^{n-1}+aqx^{n-2}+arx^{n-3}+\&c.=0$. Fiat hujusmodi productorum summa; proveniet $(a+nb)x^n+(a+\overline{n-1}b)px^{n-1}+(a+\overline{n-2}b)qx^{n-2}+(a+\overline{n-3}b)rx^{n-3}+\&c.=0$. Sed hæc summa multiplum quantitatis $(x-r)^{m-1}$ complectitur, & simul exhibet productum ex progressione qualibet arithmetica $a+nb, a+\overline{n-1}b, a+\overline{n-2}b$, &c. in æquationem datam, quæ sic una dimensione deprimitur; ergo &c. *Quod erat* &c.

512.

512. *Coroll.* 1. In hujusmodi ergo æquationibus possumus quemlibet terminum tam ex extremis, quam ex intermediis per progressionem arithmeticam eliminare, posito zero sub illo termino, quem tollere cupimus.

513. *Coroll.* 2. Si æquatio data ex earum numero, de quibus agitur, multiplum continens quantitatis $(x-r)^m (x-v)^s$ &c. per progressionem arithmeticam de more multiplicetur, una dimensione quoad radices æquales, quamvis diversi generis, deprimetur; penultimus enim suæ transformatæ terminus habet in suo coëfficiente (cui Prop. præsentis demonstratio innititur) multiplum quantitatis $(x-r)^{m-1}$ $(x-v)^{s-1}$. Idem etiam eveniet si radices invicem æquales diversi generis sine inæqualium mixtione æquatio comprehendat.

514. *Coroll.* 3. Æquatio radices æquales unius generis cum inæqualibus commixtas habens, multiplum quantitatis $(x-r)^m$ comprehendit; sed per progressionem arithmeticam depressa multiplum continet quantitatis $(x-r)^{m-1}$; ergo $(x-r)^{m-1}$ est æquationi utrique communis, adeoque maximus communis earum divisor. Hoc igitur reperto, (qui est potestas radices omnes æquationis propositæ invicem æquales una minus complectens) istæ facillime

eruentur (509.), ac proinde radices omnes æquales in propofita latentes innotefcent.

515. *Coroll.* 4. Quod fi æquatio plures habeat radices æquales diverfi generis cum inæqualibus mixtas, multiplum quantitatis $(x-t)^m (x-v)^s$ &c. continebit; quapropter fi inter datam, ipfamque uno gradu prout diximus depreffam maximus communis divifor inveniatur, hic erit $(x-t)^{m-1} (x-v)^{s-1}$ &c. $= 0$. Ifta æquatio per progreffionem pariter arithmeticam uno gradu deprimatur, atque inter utramque maximus communis divifor reperiatur, qui erit $(x-t)^{m-2} (x-v)^{s-2}$ &c. $= 0$, & ita porro, donec quotus, feu maximus communis divifor fit aliqua poteftas radices omnes æquales unius generis continens, cujus pofitæ $= 0$ quum radices haberi poffint, ad reliquas æquationis propofitæ radices æquales, quamvis diverfi generis, inveftigandas eadem methodo procedere poffumus.

516. *Coroll.* 5. Si nullum æquatio propofita, & recenfita transformata communem habeant maximum divifores, radices invicem æquales propofita non poffidebit, adeoque ex hoc dijudicari facile poterit, utrum propofita radicibus invicem æqualibus gaudeat, an contra.

Exemplum I. Sit æquatio tres radices æquales continens

$x^4 -$

$x^4 - 4x^3 - 18x^2 + 108x - 135 = 0$
$0\ ,\ 1\ ,\ 2\ ,\ 3\ ,\ 4$

$-4x^3 - 36x^2 + 324x - 540 = 0$
$3\ ,\ 2\ ,\ 1\ ,\ 0$

$-12x^3 - 72x^2 + 324x - 0 = 0$, quæ divisa per x fit $-12x^2 - 72x + 324 = 0$, sive $x^2 + 6x - 27 = 0$.
Rursus $-4x^3 - 36x^2 + 324x - 540 = 0$
$0\ ,\ 1\ ,\ 2\ ,\ 3$

$-36x^2 + 648x - 1620 = 0$, sive $x^2 - 18x + 45 = 0$. Habemus igitur duas æquationes $x^2 = 18x - 45$; $x^2 = -6x + 27$; quare $18x - 45 = -6x + 27$; seu $18x + 6x = 45 + 27$; unde elicitur $x = 3$. Ergo radices quæsitæ sunt 3, 3, 3, –5.

Exemplum II. Æquatio radices æquales homogeneas continens, quarum tamen numerum ignoro, sit
$x^6 + 5x^5 - 20x^4 - 150x^3 - 135x^2 + 513x + 810 = 0$
$6\ ,\ 5\ ,\ 4\ ,\ 3\ ,\ 2\ ,\ 1\ ,\ 0$

$6x^5 + 25x^4 - 80x^3 - 450x^2 - 270x + 513 = 0$. Tam hujus, quam æquationis propositæ maximum communem divisorem inquiro, illumque invenio, esse

$$x^3 + 9x^2 + 27x + 27 = 0, \text{ cujus radix} = -\frac{9}{3} = -3$$

(509.) sed maximus iste communis divisor omnes complectitur radices æquales æquationis propositæ una

una minus (514.); ergo quatuor sunt in proposita radices æquales singulæ $= -3$.

Exemplum III. Æquatio, in qua scimus aliunde, incognitam x duos habere valores æquales, sit

$$x^3 * - ayx + y^3$$
$$3, \ 2, \ 1, \ 0$$

$3x^2 * - ay = 0$. Hinc $y = \frac{3x^2}{a}$, & $y^3 = \frac{27x^6}{a^3}$, quibus valoribus in proposita substitutis, prodit $x^3 - 3x^3 + \frac{27x^6}{a^3} = 0$, sive $27x^3 - 2a^3 = 0$, ac tandem $x = \frac{1}{3} a\sqrt[3]{2}$. Substitue hunc valorem in æquatione $y = \frac{3x^2}{a}$, & habebis $y = \frac{1}{3} a\sqrt[3]{4}$.

517. Propositio III. *Æquationis propositæ radices invicem æquales aliter invenire.*

Resolutio. Finge novam æquationem duas radices æquales indeterminatas habentem, cujus divisores hujusmodi radices contineant. Hanc multiplica per aliam æquationem generalem, ita ut productum sit ejusdem gradus cum proposita. Facta terminorum homogeneorum comparatione, si æquationis effictæ radices per sic repertos valores obtineri possunt, patet, istas fore radices invicem æquales quæsitas. *Quod erat* &c.

Exem-

Exemplum I. Sit $x^3+3x^2-9x+5=0$. Fingo æquationem $x^2-2fx+ff=0$ radices petitas complecti. Hanc multiplico per aliam $x-g=0$, ut prodeat æquatio ejusdem gradus cum proposita, & habeo

$$\left.\begin{matrix}x^3-2f\\-g\end{matrix}\right\}x^2+ff\left.\begin{matrix}\\+2fg\end{matrix}\right\}x-ffg=0.$$

Terminis homogeneis collatis, fit (1.°) $-2f-g=3$; (2.°) $ff+2fg=-9$; (3.°) $-ffg=5$. Prima ex istis æquationibus, quas *comparativas* appello, dat $g=-2f-3$; quo valore in secunda substituto, provenit $ff-4ff-6f=3ff-6f=-9$, sive $ff+2f=3$, adeoque $f=\pm 1$. Suffice pro f valorem $+1$ tam in prima, quam in tertia æquatione comparativa (quia valor -1 est inutilis) & in utraque exurget $-g=5$. Ergo quum res bene cedat, propositæ divisores erunt $x-1=0$, $x-1$; ac proinde ipsius radices æquales 1, 1, & remanens -5.

Exemplum II. Sit $x^4-4x+3=0$. Sumo $xx-2fx+ff=0$ radices quæsitas comprehendentem. Hanc duco in quadraticam generalem $x^2-px+q=0$, ut nova emergat æquatio

$$\begin{matrix}x^4-2fx^3+ffx^2-ffpx+ffq=0\\ -px^3+2fpx^2-2fqx\\ +qx^2\end{matrix}$$

ejusdem cum proposita dimensionis. Peractis comparationibus, quatuor habemus æquationes

(1.°) $2f+p=0$. (3.°) $ffp+2fq=4$.
(2.°) $ff+2fp+q=0$. (4.°) $ffq=3$.

Ex prima fit $p=-2f$; quo valore in secunda substituto, reperitur $q=3ff$; ergo quarta evadit $3f^4=3$, adeoque $f=1$. Tertia igitur transit in $p+2q=4$. Sed $p=-2$, $q=3$, ergo fit vera tertia æquatio

— 2

$-2+6=4$; quamobrem duas habet proposita radices reales, & æquales, nempe $1, 1$, duasque imaginarias $-1\pm\sqrt{-2}$.

518. *Coroll.* Nisi vel omnes, vel saltem duæ radices comparativæ magis compositæ fiant veræ, nullis proposita constabit radicibus invicem æqualibus. Ergo ex hac quoque methodo dignosci potest, utrum nec ne proposita radices invicem æquales possideat.

CAPUT V.

DE RATIONALIBUS ÆQUATIONUM LITTERALIUM FACTORIBUS, SEU DIVISORIBUS INVENIENDIS.

519. PRopositio I. *Si factores, aut divisores rationales æquatio litteralis admittat, nonnullas methodos pro iis indagandis exhibere.*

Resolutio I. Si secundi termini portionibus invicem ductis, ultimus æquationis terminus cum suo signo prodeat, dictæ portiones, mutatis signis, erunt æquationis radices. Hoc per se patet, si enim fuerit

$$\begin{aligned} x^3 - ax^2 - abx + abc &= 0 \\ + bx^2 + acx & \\ - cx^2 + bcx & \end{aligned}$$

propter $-a \times b \times -c = abc$, qui est ultimus terminus cum suo signo, erunt $+a, -b, +c$ radices æquationis.

II.

II. Si æquatio duabus tantum litteris, una cognita, puta a, & altera incognita, velut x, conflatur, ita ut in omnibus ejusdem terminis lex homogeneorum sit servata, ponatur $a = 1$, deinde invento per methodos traditas æquationis sic reductæ, quæ numerica evadet, divisore, in hujus terminis, homogeneorum lege servata, collocentur ipsius a potestates. Sic si fuerit proposita

$$4x^5 + 16ax^4 - 22a^2x^3 - 14a^3x^2 - 56a^4x + 77a^5 = 0,$$

statuendo $a = 1$, æquatio evadet

$$4x^5 + 16x^4 - 22x^3 - 14x^2 - 56x + 77 = 0,$$

cujus alter factor seu divisor quum sit $2x^2 + 8x - 11 = 0$, alter $2x^3 - 7$, cognitam suis locis rite supplendo propositæ factores fient $2x^2 + 8ax - 11a^2$; $2x^3 - 7a^3 = 0$.

III. Si æquatio duas cognitas, ex. gr. a, b, simul cum incognita x continens habere debeat divisorem duabus litteris, puta a, x, conflatum, quoniam quantitas ista propositam dividens littera b carere potest, fiat $b = 0$; deinde tam nova quantitas hinc orta, quam propositæ residuum positis $= 0$, expurgatisque hujusmodi æquationibus, maximus earum communis divisor (29) inveniatur, qui erit unus ex propositæ divisoribus; quod enim partes, totum quoque dimetitur. Sic si fuerit

$$\begin{array}{l} x^5 - 4ax^4 + 6a^2x^3 + ab^2x^2 - 2a^2b^2x + 2a^3b^2 = 0 \\ \qquad\quad - abx^3 + 2a^2bx^2 - 2a^3bx \\ \qquad\qquad\quad - 4a^3x^2 \end{array}$$

Ponendo $b = 0$, habetur $x^5 - 4ax^4 + 6a^2x^3 - 4a^3x^2 = 0$; & residuum erit

$$-abx^3 + ab^2x^2 + 2a^2bx^2 - 2a^2b^2x - 2a^3bx + 2a^3b^2 = 0.$$

In-

Inter has duas quantitates, seu inter $x^3 - 4ax^2 + 6a^2x - 4a^3 = 0$, & $- x^3 + bx^2 + 2ax^2 - 2abx - 2a^2x + 2a^2b = 0$ inveniendus est maximus communis divisor littera b carens. Si vero talis divisor existat; erit quoque communis duabus partibus $- x^3 + 2ax^2 - 2ax$, & $bx^2 - 2abx + 2a^2b$. At hujusmodi divisor esse nequit, nisi $x^2 - 2ax + 2a^2$; ergo tento, an etiam dividat $x^3 - 4ax^2 + 6a^2x - 4a^3$; sed id perfecte succedit; ergo est quoque divisor æquationis propositæ. Profecto relinquit quotum $x^3 - 2ax^2 - abx + ab^2$.

IV. Cum una ex propositæ litteris primam dimensionem non excedit, unus tantum ex propositæ divisoribus hanc litteram comprehendet, secus illa ad majores quam prima dimensiones assurgeret contra hypothesim. Aderit igitur aliquis propositæ divisor, qui eam non continebit; quapropter separetur proposita in duas partes, quarum altera dictam litteram includat, altera minime; deinde agendo ut supra, maximus communis divisor inveniatur, qui erit factor quæsitus. Sic si fuerit

$$\begin{aligned} &x^4 - 3ax^3 - 8a^2x^2 + 18a^3x + 6a^3c = 0 \\ &\quad + cx^3 - acx^2 - 8a^2cx - 8a^4 \end{aligned}$$

maximum quæro divisorem duarum partium

$$cx^3 - acx^2 - 8a^2cx + 6a^3c,$$

$$x^4 - 3ax^3 - 8a^2x^2 + 18a^3x - 8a^4,$$

quarum aggregato proposita constat, quarumque prima tantum continet litteram c ad primam potestatem elatam; & invenio, hunc esse $x^2 + 2ax - 2a^2$, qui revera est exactus propositæ divisor, nam pro quoto exhibet $x^2 - 5ax + cx + 4a^2 - 3ac$.

Cel.

Cel. *Clairaut* præter hanc methodum aliam pro divisoribus æquationum inveniendis nitide quidem affert, ut in sua Algebra videre est; nos methodum faciliorem sic proponimus.

520. Propositio II. *Si æquatio litteralis factores, seu divisores rationales admittens, & monomium habens pro ultimo termino, gaudeat maximo productorum singulos terminos multiplicantium numero, quem exigunt factores illam constantes; factores istos invenire.*

Resolutio. Sumpto aliquo ex ultimi termini divisoribus litteralibus, nulla habita ratione signorum, inspiciatur, an ille solus, vel cum aliquo numero sine fractionum interventu multiplicet incognitam unius dimensionis, ita ut addito ultimo termino, factaque æquatione ad zero, unus prodeat propositæ divisor. Deinde advertatur, an iste vel alius litteralis divisor eodem modo multiplicet aliquo in loco incognitam duarum dimensionum, simulque reperiatur in productis potestates gradatim decrescentes multiplicantibus, quamvis cum aliis litteris commixtus, ita ut addito ultimo termino, & quantitatibus istis positis $= 0$, expurgataque æquatione, alter propositæ divisor habeatur, & ita porro. Sufficienti hujusmodi divisorum numero ad reliquos factores æquationem propositam constantes, eorumque gradus dignoscendos, reperto; examinetur, an proposita omnium dictorum productorum partialium numero (non omisso primo termino) sit prædita. Quod si eveniat, reliqui factores, cujuscumque gradus sit æquatio data, methodo supra incepta certo invenientur. Id ex æquationum genesi profluit; si enim

enim duo fuerint factores, queis proposita coalescat, alter in alterius ultimum terminum totus multiplicatur; si fuerint tres, tertius per productum, qui nascitur ex ultimis terminis duorum factorum præcedentium invicem ductis, totus multiplicatur, & sic deinceps; *Quod erat* &c.

521. *Coroll.* 1. Si æquationis datæ ultimus terminus sit polynomium, hoc in suos factores litterales primo evolvatur, deinde per hujusmodi factores divisorum inquisitio, ut supra, instituatur.

522. *Coroll.* 2. Quoad reliquas æquationes, quæ allatis conditionibus careant, tres casus dari possunt factorum, seu divisorum inventionem remorantes. Fieri enim potest, ut terminus aliquis in factoribus deficiat; vel ut memorata producta incognitæ potestates concomitantia summam aut subtractionem sint passa; vel utrumque simul; proptereaque hujusmodi productorum numerus, quem exigit propositio, non sit completus. Quum autem unusquisque factor per reliquorum ultimum terminum, ut diximus, multiplicetur, si unus, vel plures factores cum æquationis portionibus remaneat sine partium imminutione commixti (quod si non semper, saltem ut plurimum accidit), isti brevi reperientur. Ergo in quolibet casu methodus ista semper est in usum vocanda, & per eam quantitatem, quam ipsa exhibet, tentanda exacta propositæ divisio; quæ si ex voto succedat, proposita deprimetur. Hac depressa, perpendendum, quod si recensita producta sine dicta imminutione scripta forent (ut accidit cum in prima operatione duo factores invicem multiplicantur) omnes

omnes quæsiti factores, seu divisores facilius unus post alium invenirentur. Ex quo monemur, unum vel plura producta, de quibus agitur, augeri aut imminui posse, donec una saltem exurgat quantitas, quæ aliis adjecta collectionem illam sine fractionum occursu, ut exposuimus, componant ad propositæ divisionem tentandam idoneam; quod haud parvam sane in hujusmodi operationibus iterandis ac rimandis patientiam sæpius requirit.

523. *Coroll.* 3. Hinc quo pluribus æquatio litteris constat, eo probabilior evadit unius vel plurium factorum expedita investigatio; probabilior namque redditur unius vel plurium factorum integrorum cum reliquis propositæ partibus commixtio.

524. *Coroll.* 4. Si æquationis datæ termini homogeneorum legem non servent, nova littera, ut jam monuimus (336.) ad hanc legem instaurandam in proposita loca convenientia invehantur; tum reliqua ut supra peragantur; sic divisoris alicujus inventio facilior reddi potest. Sed hæc omnia sunt Exemplis illustranda.

Exemplum I. Esto

$$\begin{aligned}6x^5 - 12ax^4 - 15abx^3 - 9cd^2x^2 - 12cd^2fx + 3cd^2g^2 = 0.\\ + 8fx^4 - 16afx^3 + 4ag^2x^2 + 5abg^2x\\ - 2g^2x^3 - 20abfx^2\end{aligned}$$

Quum per nullum ex solis ultimi termini divisoribus sit incognita linearis x multiplicata, nec ullum habebit proposita divisorem linearem, nec ullum idcirco quadraticum. Ergo ex duobus factoribus, altero quadratico, altero cubico coalescit. Quare numerus productorum integrorum, non omisso primo ter-

mino, esse debet $3.4 = 12$, quia 3, & 4 esse possunt factorum propositam gignentium termini, si unam tantum litteram eorum coëfficientes habeant; quod quum in proposita reperiatur, capio ultimi termini $3cd^2g^2$ divisorem, qui solus multiplicet incognitam quadraticam x^2; at hic est cd^2; ergo juxta regulam sumo $-9cd^2x^2 - 12cd^2fx + 3cd^2g^2 = 0$; & dividendo per 3, habeo $3x^2 + 4fx - g^2 = 0$. Transeo ad incognitam cubicam x^3. Quum autem ista sit per solum g^2 ex ultimi termini divisoribus multiplicata, reperio

$$-2g^2x^3 + 4cg^2x^2 + 5abg^2x + 3cd^2g^2 = 0.$$

Divido per g^2, & obtineo

$$2x^3 - 4ax^2 - 5abx - 3cd^2 = 0.$$

Ergo duo sunt propositæ factores, nempe

$$3x^2 + 4fx - g^2;\ 2x^3 - 4ax^2 - 5abx - 3cd^2;$$

quibus invicem ductis, proposita revivifcit.

Exemplum II. Esto

$$\begin{array}{l} 2x^5 + 2cx^4 + 2acx^3 + 2a^2bx^2 - 3a^2bfx + 4a^2bgb = 0 \\ \quad + 2ax^4 - 2a^2x^3 + 2bd^2x^2 - 3bd^2fx + 4bd^2gb \\ \quad - 2bx^4 - 2bcx^3 - 2a^3x^2 + 3a^3fx - 4a^3gb \\ \quad - 3fx^4 - 2d^2x^3 - 2ad^2x^2 + 3ad^2fx - 4ad^2gb \\ \qquad\qquad - 3cfx^3 - 3acfx^2 + 4acgbx \\ \qquad\qquad - 3afx^3 + 3a^2fx^2 - 4a^2gbx \\ \qquad\qquad - 3bfx^3 + 3bcfx^2 - 4bcgbx \\ \qquad\qquad + 4gbx^3 + 3d^2fx^2 - 4d^2gbx \\ \qquad\qquad\qquad\quad + 4cgbx^2 \\ \qquad\qquad\qquad\quad + 4agbx^2 \\ \qquad\qquad\qquad\quad - 4bgbx^2 \end{array}$$

Evolvo primo ultimum terminum in factores $(a-b)$ $(-a^2-d^2)$ $(4gb)$; secundo inspicio, an aliquis ex

hisce

hisce factoribus solus multiplicet incognitam linearem x, & reperio, hunc tantummodo esse $-4a^2gb-4d^2gb$; idem perago cum incognita quadratica x^2, & invenio, hunc esse a^2b+bd^2, vel $4agb-4bgb$. Ergo propositæ factores erunt duo quadratici, & unus linearis; si enim omnes essent lineares, secundi termini portiones invicem ductæ gignerent ultimum terminum cum suo signo (519.), quod non accidit. Hinc si incognitæ coëfficientes in hisce factoribus una littera constent, numerus productorum simul cum primo termino in proposita erit vel $4.4.2=32$; vel $3.4.3=36$, quia ultimus ejusdem terminus tribus componitur factoribus $(a-b)(-a^2-d^2)(4gb)$. Recenseo igitur dicta producta una cum primo termino, & reperio numerum 36; quare factor linearis habet binomium pro ultimo termino. Interea sic omnes factores expiscor. Ultimi propositæ termini qui soli incognitam linearem x multiplicant, sunt $-4a^2gb-4d^2gb$, cum reliquis enim aliqua miscetur littera, quæ in ultimo termino non adest, quod sit norma pro reliquis casibus; hinc $-(4a^2gb+4d^2gb)$ $x+4a^2bgb-4ad^2gb=0$, sumendo solum ultimi termini portiones per incognitæ factores divisibiles, quod posthac semper animadvertatur; quamobrem $x+a-b$ erit factor linearis.

Pergo ad producta incognitæ x^2; & quoniam colligere debeo portiones, quæ per eandem quantitatem sint multiplicatæ, habeo $(a^2b+bd^2)2x^2-(a^2b+bd^2)3fx+(a^2b+bd^2)4gb=0$, seu $2x^2-3fx+4gb=0$ pro altero ex factoribus quadraticis. Idem obtineo, capiendo $-(a^3+ad^2)2x+(a^3+ad^2)3fx$

$-(a^3+ad^2)\,4gb = 0$, ſed facilius per $-2ad^2x^2 + 3ad^2fx - 4ad^2gb = 0$, vel $-2a^3x^2 + 3a^3fx - 4a^3gb = 0$.

Nunc ad alterum factorem quadraticum eſt tranſeundum; pro quo incipere nequeo, niſi a portionibus $4agbx^2 - 4bgbx^2$; ergo ſeligo $(4agb - 4bgb)x^2 + (4agb - 4bgb)cx + (4agb - 4bgb)(-a^2 - d^2)$, ſeu $x^2 + cx - a^2 - d^2$; qui factores inter ſe multiplicati propoſitam reſtituunt.

Sed videamus, an factorem cubicum continere propoſita valeat. Itaque quoniam $4gb$ multiplicat ultimum terminum, accipio $4gbx^3 + 4gb(a - b + c)x^2 + 4gb(ac - a^2 - bc - d^2)x + 4gb(a^2b + bd^2 - a^3 - ad^2) = 0$, ſeu $x^3 + (a - b + c)x^2 + (ac - a^2 - bc - d^2)x + a^2b + bd^2 - a^3 - ad^2 = 0$, qui eſt quoque propoſitæ factor pro quoto exhibens $2x^2 - 3fx + 4gb$. Eandem proſequor perquiſitionem; & quoniam excepto $4gb$, nulli alii factores ultimi termini præter $-a^2 - d^2$ multiplicant $2x^3$, iſtos pro communibus productis accipio. At prius conſidero, eſſe $(2a^2b + 2bd^2 - 2a^3 - 2ad^2)x = (-a^2 - d^2)(2a - 2b - 3f)$; & $(-3a^2bf - 3bd^2f + 3a^3f + 3ad^2f - 4a^2gb - 4a^2gb)x = (-a^2 - d^2)(-3af + 3bf + 4gb)x$; nec non $4a^2bgb + 4bd^2gb - 4a^3gb - 4a^2d^2gb = (-a^2 - d^2)(4agb - 4bgb)$; quare hinc eruo

$$\begin{array}{l} 2x^3 + 2ax^2 + 3bfx + 4agb = 0, \\ \quad - 2bx^2 - 3afx + 4bgb \\ \quad - 3fx^2 + 4gbx \end{array}$$

qui eſt alter propoſitæ factor cubicus.

Si quis autem ſcire aveat, an propoſita factorem biquadraticum comprehendat, nullus alius factor communis, qui multiplicet $2x^4$ eligi nequit, præter $a - b$, quod c, & f non inſunt in ultimo termino. Quæro igi-

igitur, quas litteras ad x^3 pertinentes multiplicet $a-b$, & invenio $2c-3f$. Transeo ad x^2 cum suis productis, & video, esse $-2a^3x^2+2a^2bx^2=(a-b)(-2a^2x^2)$; $2ad^2x^2+2bd^2x^2=(a-b)(-2a^2x^2)$; $-3acfx^2+3bcfx^2=(a-b)(-3cfx^2)$; $4agbx^2-4bgbx^2=(a-b)4gb$. Quamobrem obtineo $(a-b)(3a^2f+3d^2f+4cgb)x$, ac tandem $(a-b)(-4a^2gb-4d^2gb)$. Hinc, agendo ut supra, statuo, factorem biquadraticum quæsitum esse

$$\begin{aligned} 2x^4+2cx^3-2a^2x^2&+3a^2fx-4a^2gb=0\\ -3fx^3-2a^2x^2&+3d^2fx-4d^2gb\\ -3cfx^2&+4cgbx\\ +4gbx^2& \end{aligned}$$

per hunc enim divisa æquatio proposita dat $x+a-b$ pro quotiente.

Exemplum III. Esto $x^9+(3ab+3cd)x^7+4f^3x^6+(3a^2b^2+9abcd)x^5+12abf^3x^4+(9a^2b^2cd+a^3b^3)x^3+12a^2b^8f^3x^2+3a^3b^3cdx+4a^3b^3f^3=0$.

Statim dispicitur, hanc æquationem carere numero completo dictorum productorum; ergo tentandum, an aliquis factor indagari possit. Interea video, factorem linearem abesse, factoremque quadraticum haberi non posse, quum enim fiat $12a^2b^2f^3x^2+4a^3b^3f^3=0$, sive $3x^2+ab=0$, dividendo propositam per hanc quantitatem, nascitur fractio, nec divisio peragi potest. Transeo igitur ad factores incognitæ cubicæ; & quoniam a^2b^2cd non est factor exactus ultimi termini, sumo $a^3b^3x^3$, & separando terminos per a^3b^3 multiplicatos, habeo $a^3b^3x^3+3a^3b^3cdx+4a^3b^3f^3=0$, nimirum $x^3+3cdx+4f^3=0$, per quem tentanda est propositæ divisio. Sed interim cer-

no, plures terminos (incipiendo ab incognita sex dimensionum) esse per f^3 multiplicatos; quare hos segrego, & obtineo $4f^3x^6 + 12abf^3x^4 + 12a^2b^2f^3x^2 + 4a^3b^3f^3 = 0$; itaque dividendo per $4f^3$, remanet $x^6 + 3abx^4 + 3a^2b^2x^2 + a^3b^3 = 0$; hunc multiplico per $x^3 + 3cax + 4f^3$, & exurgit proposita.

525. *Scholion.* Si factores inventos attentius intueor, facile percipio, istos non esse duos, sed quatuor, est enim $x^6 + 3abx^4 + 3a^2b^2x^2 + a^3b^3 = (x^2+ab)(x^2+ab)(x^2+ab)$; ex quo discimus, quod si duo vel plures factores invicem æquales datam æquationem ingrediantur, non illos singillatim, sed eorum tantummodo productum invenire datum erit.

Exemplum IV. Esto

$$\begin{array}{llllll} x^5 - 4ax^4 & + 6a^2x^3 & + ab^2x^2 & - 2a^2b^2x & + 2a^3b^2 & = 0 \\ & - abx^3 & + 2a^2bx^2 & - 2a^3bx & & \\ & & - 4a^3x^2 & & & \end{array}$$

Factores lineares $x-a$, & $x-b$ non sunt exacti propositæ divisores, quare numerus 10 portionum, in quas proposita dispertitur, non est requisitus, quod etiam dignoscitur ex tribus tantum litteris a, b, x totam æquationem constantibus. Itaque omissis factoribus linearibus, gradior ad quadraticos. Hic autem reiicio $4a^3x^2 - 2a^3bx + 2a^3b^2$, quod $2x^2 - bx + b^2$ propositam ob $2x^2$ dividere nequit. Idem exequor quoad inutilem $2a^2bx^2 - 2a^3bx + 2a^3b^2$; ac tandem experior $ab^2x^2 - 2a^2b^2x + 2a^3b^2 = 0$, seu $x^2 - 2ax + 2a^2 = 0$, quæ optime cedit, pro quotiente præbens $x^3 - 2ax^2 - abx + ab^2$.

526. *Scholion.* Difficile sane foret factorem istum cubicum in prima operatione venari, at non impossi-

possibile; ceteris enim collectionibus frustra tentatis, dividendo tandem $6a^2x^3$ in duas partes $4a^2x^3$, & $2a^2x^3$, remaneret tentanda quantitas $2a^2x^3 - 4a^2x^2 - 2a^3bx + 2a^3b^2 = 0$, seu $x^3 - 2ax^2 - abx + ab^2$, quæ ad finem optatum perduceret.

Exemplum V. Esto $x^6 + bx^5 - bx^4 + x^2 + bx - b = 0$. Quoniam homogeneorum lex non est servata, æquationem sic scribo

$$x^6 + bx^5 - abx^4 + a^4x^2 + a^4bx - a^5b = 0;$$

quam statim ac considero, mihi occurrit $a^4x^2 + a^4bx - a^5b = 0$, seu $x^2 + bx - ab = 0$. Itaque tento, an hæc quantitas æquationem correctam dividat; quod quum ex voto succedat, habeo $x^4 + a^4$ pro quotiente. Reliquæ quantitates, quæ sese offerunt examinandas, nempe $-abx^4 + a^4bx - a^5b = 0$, vel $bx^5 - abx^4 + a^4bx - a^5b = 0$, postquam fuerint expurgatæ frustra tentantur. Sed sat est alterum ex divisoribus invenire, ut alter per divisionem emergat. Jam vero si ponatur $a = 1$, propositæ factores, aut divisores erunt $x^4 + 1$, & $x^2 + bx - b = 0$.

Exemplum VI. Esto tandem

$$\begin{aligned} x^4 - 3ax^3 - 8a^2x^2 + 18a^3x + 6a^3c &= 0 \\ + cx^3 - acx^2 - 8a^2cx - 8a^4 & \end{aligned}$$

Quoniam $3a - c$ non est ultimi termini divisor, statim coniicio, æquationem propositam neque factore lineari, neque cubico præditam esse, quare duplicem factorem quadraticum, cujus alter habet binomium pro ultimo termino, æquatio comprehendit; sed non habet dictorum productorum una cum primo termino numerum requisitum, qui etiam positis incognitæ coëfficientibus monomiis esse deberet $= 3$.

$= 3 \cdot 4 = 12$; ergo tentandum, an aliquis propositæ factor detegatur. Itaque ut producto incognitam quadraticam comitanti aptare possim aliquem ultimi termini divisorem, perpendo, esse $-x^2$ multiplicatum per $ac + 8a^2$; quare quum $ac + 8a^2$ non sit ultimi termini divisor, sed $3ac - 4a^2$, qui multiplicat $2a$, imaginor, esse ac imminutum, auctumque a^2 plus quam exigit factoris inquisitio, adeoque [*vi Coroll.* 2.] augeo acx^2, & imminuo $8a^2$, ita ut fiat productum $-x^2[3ac - 4a^2]$. Itidem ad eundem divisorem $3ac - 4a^2$ accommodo quartum æquationis datæ terminum $[18a^3 - 8a^2c]x$, scribens $-ax[3ac - 4a^2]$. Hinc colligo $-x^2[3ac - 4a^2] - ax[3ac - 4a^2] + 2a[3ac - 4a^2] = 0$, seu $x^2 + ax - 2a^2 = 0$. Exploro divisionem, qua exacta evadente, habeo factorem quæsitum, qui relinquit quotum $x^2 - 5ax + 4a^2$
$+ cx - 3ac$

527. *Scholion.* Si statim sumpsissem $2a^2$ pro ultimi termini factore, prodiisset $2a^2[4x^2] + 2a_2[4c - 9a]x + 2a^2[4a^2 - 3ac]$, seu $x^2 + [4c - 9a]x + 4a^2 - 3ac$, quia primus æquationis terminus habere debet unitatem pro coëfficiente; quamobrem propter binomium $4c - 9a$, quo constat secundi termini coëfficiens, factoris inventio, nisi fortuito occurreret, laborem improbum exposceret, quamvis tandem fieret Analysta voti compos. Præstat igitur viam breviorem ac faciliorem primum eligere ab ultimi termini factoribus indicatam, per quam uno saltem propositæ factore invento, alius difficilioris indaginis per divisionem innotescat.

528.

528. *Definitio.* Æquatio, quæ per aliam ordinis inferioris dividi poteſt, dicitur *reducibilis*; quæ vero diviſionem exactam reſpuit, *irreducibilis* appellatur.

529. Propoſitio III. *An æquatio data factores habeat cujuſcumque gradus numericos aut litterales, rationales aut irrationales per aliam methodum investigare.*

Reſolutio I. Ponamus, propoſitam eſſe productum duarum æquationum, quas voco *auxiliares*, videlicet

$$x^n + \alpha x^{n-1} + \beta x^{n-2} + \gamma x^{n-3} + \ldots + P = 0 \ldots (M)$$

$$x^{m-n} + Ax^{m-n-1} + Bx^{m-n-2} + Cx^{m-n-3} + \ldots + Q = 0 \ldots (N)$$

Quum productum iſtud habeat coëfficientes homologis propoſitæ coëfficientibus æquales (478.), fiant comparationes, ut coëfficientium α, β, γ, &c. A, B, C, &c. valores determinentur. Omnes iſtæ æquationes, quas ut ante *comparativas* appello tamdiu tractentur, donec nova fiat ex ipſis æquatio indeterminatas tantummodo α, β, &c. & ultimum æquationis (M) terminum P continens. Sed quia P, dum termini homologi comparantur, evadere ſemper debet unus ex diviſoribus ultimi termini æquationis propoſitæ, is pro quantitate cognita haberi poteſt. Per unum igitur aut alterum ex iſtis diviſoribus determinentur valores indeterminatarum α, β, &c. ac poſtmodum exploretur, an iſti valores ſimul cum valore P in reſpondentem æquationem auxiliarem introductus faciat, ut iſta evadat exactus propoſitæ diviſor, quod ſi accidat, æquatio erit reducibilis, & contra.

II.

II. Vel brevius. Valoribus α, β, γ &c. P, prout monuimus, inventis, per ipsos habeantur valores indeterminatarum A, B, C, &c. Q, atque hi valores in æquatione comparativa, ubi major adest ipsarum A, B, C, &c. numerus, substituatur; quod si æquatio ista evadat identica, signum est, dictos valores quæsito satisfacere; *Quod erat* &c.

Exemplum I. Finge, æquationem $x^4 + px^3 + qx^2 + rx + s = 0$, æquationes omnes biquadraticas designantem ex duabus oriri auxiliaribus indeterminatis

$$x^2 + \alpha x + \beta = 0 ; \quad x^2 + Ax + B = 0$$

Hisce invicem ductis, proveniet

$$\begin{aligned} x^4 + Ax^3 + Bx^2 &+ B\alpha x + B\beta = 0 \\ + \alpha x^3 + A\alpha x^2 &+ A\beta x \\ + \beta x^2 & \end{aligned}$$

Quoniam ista propositæ ponitur æqualis, terminis homologis conlatis, habentur æquationes comparativæ

$$(1.^a)\ A + \alpha = p ; \quad (2.^a)\ B + A\alpha + \beta = q ;$$
$$(3.^a)\ B\alpha + A\beta = r ; \quad (4.^a)\ B\beta = s.$$

Ex prima fit $A = p - \alpha$; ex quarta $B = \frac{s}{\beta}$; quibus valoribus in tertia substitutis, prodit $\frac{s\alpha}{\beta} + p\beta - \alpha\beta = r$.

Et multiplicando per β, erit $s\alpha + p\beta^2 - \alpha\beta^2 = \beta r$.

Hinc $s\alpha - \alpha\beta^2 = r\beta - p\beta^2$, ac tandem $\alpha = \frac{r\beta - p\beta^2}{s - \beta^2}$.

Quum

Quum autem sit $B = \frac{s}{\beta}$, erit β unus ex divisoribus cognitis ultimi termini s. Oportet igitur hos divisores cum utroque signo $\pm$ in formulam $\frac{r\beta - p\beta^2}{s - \beta^2}$ pro β introducere, ut valor α determinetur; qui æquationem comparativam, ubi adsunt A, & B valeant reddere identicam.

Sic si fuerit $x^4 + 2x^3 - 2x^2 - 18x - 7 = 0$, erit $p = 2$; $q = -2$; $r = -18$; $s = -7$. Sed ultimi termini divisores sunt ± 1, ± 7; ergo si pro β sumatur divisor 1, res non succedit, ut tentanti palam erit; quare si ponatur $\beta = -1$, fiet $\alpha = \frac{18 - 2}{-7 - 1} = -2$, quibus valoribus in primam æquationem auxiliarem $x^2 - \alpha x + \beta = 0$ introductis, habetur æquatio $x^2 - 2x - 1 = 0$ æquationem propositam perfecte dividens, superstite quoto $x^2 + 4x + 7$. Vel ob $\alpha = -2$, $\beta = -1$, erit $A = 4$, $B = 7$, qui valores si in secundam æquationem comparativam $A\alpha + B + \beta = q$ transferantur, fiet $-2 = -2$; quare quum ista æquatio sit identica, indicat, valores adhibitos ad problematis solutionem perducere. Itaque factis substitutionibus, duæ æquationes auxiliares rursus ut supra reperientur. Similiter si fuerit $x^4 + 2bx^3 + b^2x^2 * - a^3b = 0$, fiet $p = 2b$; $q = b^2$; $r = 0$; $s = -a^3b$. Hinc $\alpha = \frac{-p\beta^2}{s - \beta^2}$. Quia vero pro β accipi debet divisor

ali-

aliquis ultimi termini $-a^3b$, sumo (ob æquationis ultimum terminum β secundi ordinis) divisores secundi ordinis $\pm a^2, \pm ab, \pm a\sqrt{ab}$. Sed quia reliqui reperiuntur inutiles, retineo $\beta = a\sqrt{ab}$ tam positivum, quam negativum, & in utroque casu invenio

$\alpha = \frac{-p\beta^2}{s-\beta^2} = b$. Substituo hunc valorem in æquationibus comparativis generalibus; & quia $A = b$, $B = \pm a\sqrt{ab}$, pro æquatione secunda $A\alpha + B + \beta = q$ obtineo æquationem identicam $b^2 = b^2$, quæ demonstrat, divisores trinomios quæsitos esse $x^2 + bx + a\sqrt{ab}$; $x^2 + bx - a\sqrt{ab}$; qui profecto invicem ducti propositam restituunt.

530. *Scholion.* Dum alicujus æquationis gradus paris factores quadraticos venamur, fieri potest, ut duo, vel plures ex istis eadem prædici sint quantitate cum eodem signo in ultimo termino; tunc autem per hanc methodum ad inutilem perveniemus

æquationem $\alpha = \frac{0}{0}$, quamvis verus ultimi termini propositæ divisor eligatur. Proponatur enim $x^4 - 11x^3 + 34x^2 - 33x + 9 = 0$. Erit $p = -11$, $q = 34$, $r = -33$, $s = 9$. Sumatur $\beta = 3$, qui est

numeri 9 genuinus divisor. Reperietur $\alpha = -\frac{99+99}{9-9}$

$= \frac{0}{0}$, quod nihil præbet; ergo alius quantitatis α

va-

valor inquirendus. Itaque quoniam vidimus, esse $A + \alpha = p$; $A\alpha + B + \beta = q$; $B\alpha + A\beta = r$; $B\beta = s$, per secundam & quartam ex his æquationibus eiicio quantitatem B, atque habeo $s = q\beta - A\alpha\beta - \beta^2$. Per hanc præterea & primam æquationem elimino A, ut fit $s = q\beta - p\alpha\beta + \alpha^2\beta - \beta^2$; & ordinando per α, obtineo $\alpha^2 - p\alpha - \beta + q = \frac{s}{\beta}$;

ex quo tandem $\alpha = \frac{1}{2}p \pm \sqrt{\frac{s}{\beta} - q + \beta + \frac{p^2}{4}}$; quare retento $\beta = 3$, factisque substitutionibus, obveniet $\alpha = -\frac{11}{2} \pm \sqrt{3 - 34 + 3 + \frac{121}{4}} = -$

$\frac{11 \pm 3}{2}$. Ergo duos α proferet valores, nimirum -4, & -7, quibus in æquationes auxiliares inventis, proveniunt duo factores $x^2 - 4x + 3$; $x^2 - 7x + 3$ propositam gignentes.

Idem quoque usuvenit in secundi gradus æquationibus auxiliaribus determinandis, cum æquatio proposita est litteralis. Esto namque $x^4 - (2ab + a^2)x^2 + a^2b^2$. In hac habemus $p = 0$; $q = -(2ab + a^2)$; $r = 0$; $s = a^2b^2$. Sumpto igitur $\beta = -ab$, substitutisque in formula $\alpha = \frac{r\beta - p\beta^2}{s - \beta^2}$ valoribus fiet

$\alpha =$.

$\alpha = \frac{0}{0}$. Ergo ſubrogata alia formula $\alpha = \frac{1}{2}p \pm \sqrt{\frac{s}{\beta} - q + \beta + \frac{1}{4}p^2}$, prodit $\alpha = \pm a$. Hinc factores quadratici auxiliares quæſiti ſunt $x^2 + ax - ab$; $x^2 - ax - ab$; eorum enim productum æquationem datam reſtituit. Quod ſi ultimi termini æquales factorum ſignis diverſis gaudeant, tunc prima quoque formula $\alpha = \frac{r\beta - p\beta^2}{s - \beta^2}$ illis inveniendis inſervit. Sic æquationis $x^4 - a^2x^2 + 2a^2bx - a^2b^2 = 0$ factores $x^2 - ax + ab$; $x^2 + ax - ab$ per utramque formulam inveniuntur. Ex quo patet, ſecundam hanc formulam generalem eſſe primæ anteferendam.

Exemplum II. Exprimat $x^5 + px^4 + qx^3 + rx^2 + sx + t = 0$ æquationem quamcumque quinti ordinis genitam ex multiplicatione duarum æquationum, quarum altera ſecundi, altera tertii gradus, nempe $x^2 + \alpha x + \beta = 0$; $x^3 + Ax^2 + Bx + C = 0$. Quoniam harum productum

$$\begin{array}{l} x^5 + \alpha x^4 + Bx^3 + Cx^2 + C\alpha x + C\beta = 0 \\ \quad + Ax^4 + \beta x^3 + A\beta x^2 + B\beta x \\ \qquad + A\alpha x^3 + B\alpha x^2 \end{array}$$

eſt propoſitæ [*per hyp.*] æquale, æquationes comparativæ erunt quinque ſequentes

(1.ª) $A + \alpha = p$; (2.ª) $A\alpha + B + \beta = q$;
(3.ª) $A\beta + B\alpha + C = r$; (4.ª) $B\beta + C\alpha = s$;
(5.ª) $C\beta = t$.

Ex

Ex prima fit $A = p - \alpha$, ex secunda $A = \frac{q - B - \beta}{\alpha}$;

quare $p - \alpha = \frac{q - B - \beta}{\alpha}$; & $p\alpha - \alpha\alpha = q - B - \beta$;
ex quo $B = \alpha\alpha - p\alpha + q - \beta$. Sed ex quarta habetur $B = \frac{s - C\alpha}{\beta}$, seu $B = \frac{s}{\beta} - \frac{\alpha t}{\beta^2}$ ob $C = \frac{t}{\beta}$

ex quinta; ergo erit $\alpha\alpha - p\alpha + q - \beta = \frac{s}{\beta} - \frac{\alpha t}{\beta^2}$; &

ordinando æquationem per α, proveniet

$$\begin{aligned} \alpha\alpha - p\alpha + q &= 0 \ldots\ldots. (R) \\ + \frac{t\alpha}{\beta^2} - \beta & \\ - \frac{s}{\beta} & \end{aligned}$$

Ut igitur valor x innotescat, unus ex divisoribus ultimi termini s debet prius determinari, qui nuper recensitas conditiones adimpleat. Sic si fuerit $x^5 + 3ax^4 + (2b^2 + 2a^2)\, x^3 + (3ab^2 - d^3)\, x^2 + (b^4 - 2ad^3)\, x - b^2d^3 = 0$, erit $p = 3a$; $q = 2b^2 + 2a^2$; $r = 3ab^2 - d^3$; $s = b^4 - 2ad^3$; $t = -b^2d^3$. Fiat substitutio in æquatione (R), orietur

$\alpha\alpha -$

$$\left.\begin{array}{l} \alpha x - 3ax + 2b^2 + 2a^2 \\ -\dfrac{b^2d^3x}{\beta^2} - \beta \\ \qquad -\dfrac{b^4+2ad^3}{\beta} \end{array}\right\} = 0$$

Hic autem in secundo termino habemus $-\frac{b^2d^3}{\beta^2}$; ergo β esse debet unus ex factoribus ultimi termini $-b^2d^3$; quamobrem pro prima æquatione auxiliari quadratica sumendus est divisor secundi ordinis, nempe $\pm b^2$, vel $\pm d^2$, vel $\pm bd$. Sumo b^2, eritque

$$\left.\begin{array}{l} \alpha x - 3ax + 2a^2 \\ -\dfrac{d^3x}{b^2} + \dfrac{2ad^3}{b^2} \end{array}\right\} = 0$$

& extracta prius radice quadrata, tum facta ad eundem denominatorem reductione, prodit

$$\alpha = \frac{3ab^2+d^3}{2b^2} \pm \frac{\sqrt{(a^2b^4 - 2ab^2d^3 + d^6)}}{2b^2}, \text{ sive}$$

$$\alpha = \frac{3ab^2+d^3}{2b^2} \pm \frac{(ab^2-d^3)}{2b^2}$$

Duos igitur α habere potest valores, nimirum $2a$, & $a +$

$a + \frac{d^3}{b^2}$. Capio primum $\alpha = 2a$; & quoniam $\beta = b^2$, erit $A = p - \alpha = a$; $B = \frac{s}{\beta} - \frac{\alpha t}{\beta^2} = b^2$; $C = \frac{t}{\beta} = -d^3$; quibus valoribus in tertiam æquationem comparativam $A\beta + B\alpha + C = r$ introductis, fit æquatio identica $3ab^2 - d^3 = 3ab^2 - d^3$. Rejecto ergo valore $\alpha = a + \frac{d^3}{b^2}$, solitas efficio substitutiones, & obtineo $x^2 + 2ax + b^2 = 0$ pro prima æquatione auxiliari, & $x^3 + ax^2 + b^2x - d^3 = 0$ pro secunda quæ reapse inter se multiplicatæ propositam suppeditant. Ceterum unus ex hisce factoribus non ne statim reperitur per ea, quæ docuimus in Prop. præced., sumendo $-d^3x^2 - 2ad^3x - b^2d^3 = 0$, seu $x^2 + 2ax + b^2 = 0$?

Exemplum III. Æquatio omnes sexti gradus æquationes repræsentans

$$x^6 + px^5 + qx^4 + rx^3 + sx^2 + tx + v = 0$$

concipiatur componi ex duabus auxiliaribus, altera secundi gradus $x^2 + \alpha x + \beta = 0$, altera quarti gradus $x^4 + Ax^3 + Bx^2 + Cx + D = 0$, quæ invicem ductæ præbent

$$\begin{array}{l} x^6 + Ax^5 + Bx^4 + Cx^3 + Dx^2 + D\alpha x + D\beta = 0 \\ \quad + \alpha x^5 + A\alpha x^4 + B\alpha x^3 + C\alpha x^2 + C\beta x \\ \qquad\qquad + \beta x^4 + A\beta x^3 + B\beta x^2 \end{array}$$

Terminis igitur, homologis comparatis, habentur sex æquationes comparativæ, videlicet

(1.a) $A+\alpha=p$; (2.a) $A\alpha+B+\beta=q$, (3.a) $A\beta+B\alpha+C=r$; (4.a) $B\beta+C\alpha+D=s$; (5.a) $C\beta+D\alpha=t$; (6.a) $D\beta=v$.

Hinc $A=p-\alpha$; quo valore in secunda suffecto, prodit $B+p\alpha-\alpha\alpha+\beta=q$; ex quo $B=q-p\alpha+\alpha\alpha-\beta$. At in sexta stat $D\beta=v$, adeoque $D=\frac{v}{\beta}$; ergo hoc valore in quinta æquatione subrogato, oritur $\frac{\alpha v}{\beta}+C\beta=t$, unde $C=\frac{t}{\beta}-\frac{\alpha v}{\beta^2}$. Omnes isti valores sufficiantur in quarta æquatione; proveniet $\frac{v}{\beta}+\frac{t\alpha}{\beta}-\frac{\alpha^2 v}{\beta^2}+q\beta-p\alpha\beta+\alpha^2\beta-\beta^2=s$; & reducendo $\frac{v}{\beta}+\frac{t\alpha}{\beta}$ ad eundem denominatorem, fiet $\frac{v\beta+t\alpha\beta-\alpha^2 v}{\beta^2}+q\beta-p\alpha\beta+\alpha^2\beta-\beta^2=s$. Multiplica per β^2, eritque $v\beta+t\alpha\beta-\alpha^2 v+q\beta^3-p\alpha\beta^3+\alpha^2\beta^3-\beta^4=s\beta^2$, & transponendo, habetur $\alpha^2\beta^3-\alpha^2 v=p\alpha\beta^3-t\alpha\beta-v\beta-q\beta^3+\beta^4+s\beta^2$. Denique dividendo per β^3-v, & ordinando æquationem per α exurget

$$\alpha^2-\frac{p\alpha\beta^3+t\alpha\beta+v\beta+q\beta^3-\beta^4-s\beta^2}{\beta^3-v}=0\ldots.(S)$$

Itaque quum ob $D=\frac{v}{\beta}$ sit β aliquis ex divisoribus ul-

ultimi termini æquationis propositæ; singuli isti divisores in hac postrema æquatione successive substituantur ad indagandum de more valorem α, & reliqua ut ante peragantur. Sic si fuerit
$x^6 - 13cx^5 + 45c^2x^4 - 71c^3x^3 + 57c^4x^2 - 16c^5x + 2c^6 = 0$, erit $p = -13c$; $q = 45c^2$; $r = -71c^3$; $s = 57c^4$; $t = -16c^5$; $v = 2c^6$. Jam vero quum β dividere debeat $2c^6$, & pro divisoris β substitutione in æquatione auxiliari quadratica requirantur divisores secundi ordinis, pro β sumo $\pm cc, \pm 2cc, \pm cc\sqrt{2}$. Sed quia $\pm cc$, & $cc\sqrt{2}$ inutiles sum expertus, retenta $\beta = 2cc$, æquatio (S) evadit $\alpha^2 + 12c\alpha + 20c^2 = 0$, & radicem quadratam extrahendo, duo ipsius α prodeunt valores, nempe $-2c$, & $-10c$; at hoc ultimo quippe inutili rejecto, utor altero $\alpha = -2c$, atque habeo $A = -11c$; $B = -21c^2$; $C = -7c^3$; $D = c^4$; quibus valoribus in æquatione comparativâ tertia $A\beta + B\alpha + C = r$ suffectis, exurgit æquatio identica $-71c^3 = -71c^3$. Æquationes igitur auxiliares sunt $x^2 - 2cx + 2cc = 0$; $x^4 - 11cx^3 + 21c^2x^2 - 7c^3x + c^4 = 0$, quibus invicem ductis proposita reviviscit.

Si facta fuisset littera $c = 1$, ut supra monuimus, æquationis $x^6 - 13x^5 + 45x^4 - 71x^3 + 57x^2 - 16x + 2 = 0$ factores prodiissent, alter $x^2 - 2x + 2$, alter $x^4 - 11x^3 + 21x^2 - 7x + 1$, quod regulam citatam confirmat.

Exemplum IV. Fieri aliquando potest, ut æquatio sexti gradus per quadraticam, ideoque per biquadraticam æquationem dividi nequeat, quare tunc tentandum, an per duas cubicas sit reducibilis. Quamobrem esto rursus æquatio

$x^6+px^5+qx^4+rx^3+sx^2+tx+v=0$,
quam imaginemur componi ex duabus cubicis
$x^3+\alpha x^2+\beta x+\gamma=0$; $x^3+Ax^2+Bx+C=0$.
ex quarum multiplicatione consurgat

$$\begin{array}{llllllll} x^6 & +Ax^5 & +Bx^4 & +Cx^3 & +C\alpha x^2 & +C\beta x & +C\gamma & =0 \\ & +\alpha x^5 & +A\alpha x^4 & +B\alpha x^3 & +B\beta x^2 & +B\gamma x & & \\ & & +\beta x^4 & +A\beta x^3 & +A\gamma x^2 & & & \\ & & & +\gamma x^3 & & & & \end{array}$$

Ex terminorum homogeneorum collatione sequentes dimanant æquationes comparativæ
(1.a) $A+\alpha=p$; (2.a) $A\alpha+B+\beta=q$; (3.a) $A\beta+B\alpha+C+\gamma=r$; (4.a) $A\gamma+B\beta+C\alpha=s$; (5.a) $B\gamma+C\beta=t$; (6.a) $C\gamma=v$. Quum prima præbeat $A=p-\alpha$, erit $A\alpha=p\alpha-\alpha\alpha$, quo valore in secunda æquatione substituto, fit $B+p\alpha-\alpha\alpha+\beta=q$, seu $B=\alpha\alpha-p\alpha-\beta+q$. Et multiplicando per γ, oritur $B\gamma=\alpha\alpha\gamma-p\alpha\gamma-\beta\gamma+q\gamma$. At ex quinta habetur $B\gamma=t-C\beta$; ergo $\alpha\alpha\gamma-p\alpha\gamma-\beta\gamma+q\gamma=t-C\beta$, qua æquatione ducta in γ, prodit $\alpha\alpha\gamma\gamma-p\alpha\gamma\gamma-\beta\gamma\gamma+q\gamma\gamma=t\gamma-C\beta\gamma$. Est autem per sextam $C\gamma=v$; ergo substituendo provenit $\alpha\alpha\gamma\gamma-p\alpha\gamma\gamma-\beta\gamma\gamma+q\gamma\gamma=t\gamma-v\beta$, ex quo eruitur $\beta=$

$$\frac{\alpha\alpha\gamma\gamma-p\alpha\gamma\gamma+q\gamma\gamma-t\gamma}{\gamma^2-v}.$$

Ut alter ipsius β valor inveniatur, æquatio tertia $A\beta+B\alpha+C+\gamma=r$ ducatur in γ; unde $A\beta\gamma+B\alpha\gamma+C\gamma+\gamma\gamma=r\gamma$. Sed per sextam $C\gamma=v$; ergo $A\beta\gamma+B\alpha\gamma+v+\gamma\gamma=r\gamma$, seu $B\alpha\gamma=r\gamma-v-A\beta\gamma-\gamma\gamma$. Est autem per secundam $B=q-A\alpha-\beta$; ergo multiplicando per $\alpha\gamma$, prodit $B\alpha\gamma=$

$q\alpha\gamma$

$q\alpha\gamma - A\alpha\alpha\gamma - \alpha\beta\gamma$; hinc $r\gamma - v - A\beta\gamma - \gamma\gamma = q\alpha\gamma - A\alpha\alpha\gamma - \alpha\beta\gamma$. Suffice in locum A ejus valorem $p - \alpha$. Fiet $r\gamma - v - p\beta\gamma + \alpha\beta\gamma - \gamma\gamma = q\alpha\gamma - p\alpha\alpha\gamma + \alpha^3\gamma - \alpha\beta\gamma$, sive $2\alpha\beta\gamma - p\beta\gamma = \alpha^3\gamma - p\alpha^2\gamma + q\alpha\gamma + \gamma^2 - r\gamma + v$; ex quo

$$\beta = \frac{\alpha^3\gamma - p\alpha^2\gamma + q\alpha\gamma + \gamma^2 - r\gamma + v}{2\alpha\gamma - p\gamma}.$$

Ex duobus igitur ipsius β valoribus comparatis habetur

$$\frac{\alpha^2\gamma^2 - p\alpha\gamma^2 + q\gamma^2 - r\gamma}{\gamma^2 - v} = \frac{\alpha^3\gamma - p\alpha^2\gamma + q\alpha\gamma + \gamma^2 - r\gamma + v}{2\alpha\gamma - p\gamma};$$

factaque ad eundem denominatorem reductione, & æquatione ordinata per α, tandem nanciscimur

$$\alpha^3 - \frac{2p\gamma^3\alpha^2 - pv\gamma\alpha^2 - 2r\gamma^2\alpha + p^2\gamma^3\alpha + q\gamma^3\alpha + qv\gamma\alpha}{\gamma^3 + v\gamma} -$$

$$-\frac{pq\gamma^3 + pr\gamma^2 - \gamma^4 + r\gamma^3 - rv\gamma + v^2}{\gamma^3 + v\gamma} \ldots\ldots\ldots (T);$$

Ut igitur habeatur valor ipsius α, determinari ut supra prius debet γ, unus nempe ex divisoribus ultimi termini æquationis propositæ. Sic si fuerit $x^6 - 8cx^5 + 13c^2x^4 - 23c^3x^3 + 10c^4x^2 - 7c^5x - 12c^6 = 0$, erit $p = -8c$; $q = 13c^2$; $r = -23c^3$; $s = 10c^4$; $t = -7c^5$; $v = -12c^6$; hujus $-12c^6$ divisores sumendi (ob æquationes cubicas, de quibus agitur) erunt $\pm c^3, \pm 2c^3, \pm 3c^3, \pm 4c^3, \pm 6c^3, \pm 12c^3$. Tentetur jam $\gamma = + c^3$, & in æquatione (T) subrogentur valores p, q, r, &c. fiet

 $\alpha^3 +$

$$\alpha^3 + \frac{16c^{10}\alpha^2 - 96c^{10}\alpha^2 + 14c^{11}\alpha + 64c^{11}\alpha + 13c^{11}\alpha - 156c^{11}\alpha - 4c^{12}}{-11c^9}$$

$$= 0, \text{ seu } \alpha^3 + \frac{80c\alpha^2 + 65c^2\alpha - 4c^3}{11} = 0.$$

Ad eliminandam fractionem ponatur $\alpha = \frac{y}{11}$; prodibit $y^3 + 80cy^2 + 715c^2y - 484c^3 = 0$, cujus quum $y + 11c$ sit divisor exactus, habemus $y = -11c$, adeo-

adeoque $\alpha = \frac{\gamma}{11} = -c$. Substituto igitur $-c$ pro x, & $+ c^3$ pro γ in æquatione

$$\beta = \frac{\alpha^2\gamma^2 - px\gamma^2 + q\gamma^2 - r\gamma}{\gamma^2 - v}, \text{ fit}$$

$$\frac{c^2.c^6 - 8c^2.c^6 + 13c^2.c^6 + 7c^5.c^3}{c^6 + 12c^6} = \frac{13c^2}{13} = c^2$$

Quapropter ob $\alpha = -c$, $\beta = c^2$, $\gamma = c^3$, erit $A = -8c + c = -7c$; $B = c^2 - 8c^2 - c^2 + 13c^2 = 5c^2$;

$C = -\frac{12c^6}{c^3} = -12c^3$; quibus valoribus in tertiam æquationem comparativam $A\beta + Bx + C + \gamma = r$ suffectis, fit æquatio identica $-23c^3 = -23c^3$. Ergo prima æquatio auxiliaris $x^3 + \alpha x^2 + \beta x + \gamma = 0$ evadit $x^3 - cx^2 + c^2x + c^3 = 0$, & altera $x^3 + Ax^2 + Bx + C = 0$ fit $x^3 - 7cx^2 + 5c^2x - 12c^3 = 0$, ut ex earum multiplicatione, quæ propositam restituit, fit manifestum.

531. *Scholion.* Hæc methodus, quæ tentando progreditur, si in æquationibus graduum superiorum adhibeatur, majoribus semper obviam ibit difficultatibus, ita ut tandem quemvis ab incepto deterreat; si enim æquatio gradus m resolvenda foret in duas, quarum altera sit gradus $m-n$, altera gradus n; in altera ex æquationibus auxiliaribus indeterminatarum numerus erit $m-n$, in altera n; quare pronum est inspicere, difficultatem eo plus augeri,

quo major sit numerus m, quia semper occurrit numerus m indeterminatarum, quarum valores nisi per æquationes implicatissimas indagari nequeunt. Verum non semper sic agendo finem optatum in casibus etiam, qui faciliores apparent, adsequimur, possumus enim in æquationes incidere, quæ sint per methodos cognitas irreducibiles; at si aliquibus gaudeant conditionibus, in æquationes, seu trinomia secundi gradus resolvi poterunt. Itaque ut istas exponamus, sit.

532. *Definitio.* Æquationem *convertibilem* illam appello, in qua

(1.°) Primus terminus, qui esse debet gradus semper paris, sola incognita, puta x^m, constet; & ultimus sit quantitas cognita ad eundem gradum evecta, veluti a^m.

(2.°) Terminus quilibet jacens inter medium, & ultimum per quantitatis a^m congruum factorem divisus, eodem tum signo, tum coëfficiente afficiatur, quo terminus respondens jacens inter medium, & primum.

Sic æquationes $x^4+bx^3+c^2x^2+a^2bx+a^4=0$; $x^6-bx^5+c^3x^3-a^4bx+a^6=0$ sunt convertibiles., quod dividendo primæ ultimum terminum per quantitatis a^4 factorem a^2, emergit $+bx$, qui eodem gaudet tum signo tum coëfficienti æque ac terminus secundus. In secunda vero æquatione dividendo terminum $-a^4bx$ per a^4, qui est factor quantitatis a^6, pro-

prodit $-bx$ eodem signo, & coëfficiente præditus cum termino respondente $-bx^5$.

533. *Coroll.* Ergo si quis terminus in æquatione convertibili deficiat, suus quoque terminus respondens deficere debet. Ex. gr. si secundus terminus evanescat, evanescere quoque debet penultimus, & sic de ceteris; quamobrem inter convertibiles erit quoque adnumeranda æquatio $x^m \pm a^m = 0$; in hac enim termini omnes intermedii accipi possunt veluti per zero multiplicati.

534. Propositio IV. *Æquationes convertibiles resolvere.*

Resolutio. Formetur æquatio secundi gradus, nempe $x^2 + fx + a^2 = 0$, in qua f est quantitas indeterminata, cujusque ultimus terminus a^2 sit quadratum radicis quantitatis a^m ultimum propositæ terminum repræsentantis. Tum formetur ad arbitrium alia æquatio convertibilis duobus gradibus præ data inferior, cujus pariter coëfficientes sint indeterminati, excepto ultimo termino a^{m-2}, qui cum ultimo propositæ idem esse debet, sed duobus gradibus inferior. Duabus istis æquationibus invicem ductis, tertia orietur, quæ ejusdem erit gradus ac proposita, quæque cum ea consideretur identica. Quare coëfficientibus homogeneis, ut nuper egimus, comparatis usque ad terminum medium inclusive (reliquis enim terminis collatis, eædem redeunt æquationes), omnes indeterminatæ adsumptæ præter f eli-

mi-

minentur. Hinc prodibit æquatio, quæ dabitur per f & conſtantes; qua ut moris eſt reſoluta ad valores omnes ipſius f (quorum numerus erit $\frac{m}{2}$) eliciendos, valores iſti in trinomium x^2+fx+a^2 ſucceſſive introducti exhibebunt trinomia ſecundi gradus, puta x^2+Ax+a^2; x^2+Bx+a^2; x^2+Cx+a^2, &c. in quæ convertibilis propoſita reſolvitur. *Quod erat* &c.

Exemplum I. Sit $x^4+3bx^3+5c^2x^2+3a^2bx+a^4=0$.

Sumo æquationem convertibilem quadraticam $xx+fx+aa=0$, quam multiplico per trinomium $xx+gx+aa$, ut exurgat æquatio convertibilis

$$\begin{array}{l} x^4+fx^3+2a^2x^2+a^2gx+a^4=0 \\ \quad\;\; +gx^3+fgx^2\;\; +a^2fx \end{array}$$

Terminorum comparatio præbet $f+g=3b$, ſeu $f=3b-g$; $2a^2+fg=5c^2$, ſeu valorem ipſius f ſubſtituendo, habetur $2a^2+3bg-g^2=5c^2$, cujus radices $g=\frac{3}{2}b^2\pm\sqrt{\frac{9}{4}b^2+2a^2-5c^2}$. Hos valores ſufficio in trinomio $xx+gx+aa$, & obtineo

$$xx+\left(\frac{3}{2}b^2+\sqrt{\frac{9}{4}b^2+2a^2-5c^2}\right)x+a^2=0$$

$$xx+\left(\frac{3}{2}b^2-\sqrt{\frac{9}{4}b^2+2a^2-5c^2}\right)x+a^2=0$$

qui ſunt propoſitæ factores.

Exem-

Exemplum II. Sit æquatio convertibilis
$x^6 + abx^4 + c^3x^3 + a^3bx^2 + a^6 = 0$.
Efficio binomium quadraticum $xx + fx + aa = 0$, quo in æquationem convertibilem biquadraticam $x^4 + gx^3 + b^2x^2 + a^2gx + a^4 = 0$ ducto, provenit

$$\begin{array}{l} x^6 + fx^5 + a^2x^4 + 2a^2gx^3 \\ \quad + gx^5 + fgx^4 + fb^2x^3 \\ \qquad + b^2x^4 \end{array} \quad \&c. = 0$$

Facta igitur consueta coëfficientium homologorum comparatione, prodit $f + g = 0$, seu $g = -f$; $a^2 + fg + b^2 = ab$, seu $b^2 = ab - fg - a^2$; $2a^2g + fb^2 = c^3$, seu $2a^2g + abf - f^2g - a^2f = c^3$, valorem ipsius b^2 substituendo. In hac ultima æquatione colloco valorem g, ex quo fit $-2a^2f + abf + f^3 - a^2f = c^3$, sive $f^3 + (ab - 3a^2)f - c^3 = 0$. Hinc tres eliciuntur valores, nempe

$$f = \sqrt[3]{\frac{c^3}{2} + \sqrt{\frac{c^6}{4} + \frac{a^3b^3 - 9a^4b^2 + 27a^5b - 27a^6}{27}}} + \sqrt[3]{\frac{c^3}{2} - \sqrt{\frac{c^6}{4} + \frac{a^3b^3 - 9a^4b^2 + 27a^5b - 27a^6}{27}}}$$

duoque reliqui, qui eodem constant valore ipsius f per duas quantitates imaginarias, ut opportuniori loco docebitur, multiplicato. Itaque si prima ex hujusmodi radicibus vocetur p, secunda q, tertia r, tria trinomia, queis proposita componitur, erunt $x^2 + px + a^2 = 0$; $x^2 + qx + a^2 = 0$; $x^2 + rx + a^2 = 0$.

Exem-

Exemplum III. Sit $x^6 + a^6 = 0$. Post solitam efformatam æquationem quadraticam $x^2 + fx + a^2 = 0$, per hanc multiplica convertibilem biquadraticam $x^4 + gx^3 + hx^2 + a^2gx + a^4 = 0$, & habeo

$$\begin{matrix} x^6 + gx^5 + hx^4 + 2a^2gx^3 \\ + fx^5 + fgx^4 + fhx^3 \\ + a^2x^4 \end{matrix} \quad \&c. = 0$$

Igitur de more comparando, tres occurrunt æquationes $f + g = 0$; $h + fg + a^2 = 0$; $2a^2g + fh = 0$. Ex prima fit $g = -f$, quo valore in duabus reliquis suffecto, prodeunt $h - f^2 + a^2 = 0$; $-2a^2f + fh = 0$. Sed $h = f^2 - a^2$; ergo hoc valore in ultima $-2a^2f + fh = 0$ subrogato, oritur $f^3 - 3a^2f = 0$; unde tres eliciuntur ipsius f valores, nempe $f = 0$, $f = a\sqrt{3}$, $f = -a\sqrt{3}$; quibus valoribus in trinomium $x^2 + fx + a^2 = 0$ invectis, tres exurgunt æquationes quadraticæ $xx + aa = 0$; $xx + ax\sqrt{3} + aa = 0$; $xx - ax\sqrt{3} + aa = 0$, in quas proposita resolvi poterit.

535. *Scholion.* (1.°) Quamquam æquationes convertibiles esse debent gradus paris, æquatio $x^m \pm a^m$, quæ ad illas, ut diximus, refertur, gradus impares admittere potest. Consideremus primo æquationem $x^m + a^m = 0$. Si m sit numerus impar, ista erit productum binomii $x + a$ in æquationem $x^{m-1} - ax^{m-2} + a^2x^{m-3} - a^3x^{m-4} + \&c. \ldots\ldots + a^{m-1} = 0$, ut

ut dividendo $x^m + a^m$ per $x + a$ verum reperitur. Hæc autem æquatio quum sit ex convertibilium genere, in sua trinomia quadratica resolvetur, adeoque $x^m + a^m$ partim in binomium $x + a$, partim in trinomia resolvi poterit.–Ubi animadvertendum, æquationem convertibilem, quæ formanda est, ut habeatur productum ex ipsa in trinomium $x^2 + fx + a^2$, esse debere gradus $m - 3$, quod $m - 3 + 2 = m - 1$ efficit gradum æqualem dictæ æquationi $x^{m-1} - ax^{m-2} + a^2x^{m-3} - \&c. + a^{m-1} = 0$.

(2.°) Eodem modo si in æquatione $x^m - a^m = 0$ est m numerus impar, ipsa oritur ex producto binomii $x - a$ in $x^{m-1} + ax^{m-2} + a^2x^{m-3} + a^3x^{m-4} + \&c. \ldots\ldots + a^{m-1} = 0$, ut per divisionem binomii $x^m - a^m$ per $x - a$ fit manifestum; quare operatio ut in præcedenti $x^m + a^m$ instituetur.

(3.°) Quod si æquationis $x^m - a^m = 0$ exponens m est numerus par, resolvatur æquatio in duo binomia $x^{\frac{m}{2}} + a^{\frac{m}{2}}$; $x^{\frac{m}{2}} - a^{\frac{m}{2}}$; nam duo ista binomia in-

invicem ducta præbent $x^m - a^m$. Horum autem binomiorum alter $x^{\frac{m}{2}} + a^{\frac{m}{2}}$, in quo $\frac{m}{2}$ est numerus par, protinus tractatur ad normam æquationum convertibilium; cum vero $\frac{m}{2}$ est numerus impar, tractatur, prout hic supra (n. 1.°) docuimus. Alter $x^{\frac{m}{2}} - a^{\frac{m}{2}}$ resolvitur in duo binomia $x^{\frac{m}{4}} + a^{\frac{m}{4}}$; $x^{\frac{m}{4}} - a^{\frac{m}{4}}$; quorum primum, quicumque sit numerus $\frac{m}{4}$ tractatur, ut in cit. num. 1.°, secundus resolvitur in duo binomia $x^{\frac{m}{8}} + a^{\frac{m}{8}}$; $x^{\frac{m}{8}} - a^{\frac{m}{8}}$; & sic deinceps, donec quantitatis $x - a$ termini habeant pro exponente numerum imparem, tunc enim resolutio, ut ante (n. 2.°) perficietur. Qui resolutiones hasce videre cupit, adeat *Gabrielem Manfredum*, Analystam doctissimum, in *Instit. Bonon.* T. I. Interea mihi liceat perbrevem methodum pro nonnullis casibus hisce affinibus hic afferre.

536. Propositio V. *Binomii* $a^m + b^n$, *cujus termini ad potestates quaslibet etiam diversas, perfectas aut imperfectas, positivas aut negativas, reales aut imaginarias sint evecti, plura factorum paria invenire.*

Resolutio. Finge, factores, de quibus agitur, esse $Aa^{\frac{m}{2}} + b^{\frac{n}{2}}$; $a^{\frac{m}{2}} + Bb^{\frac{n}{2}}$; in quibus A, B sunt indeterminatæ. Factoribus istis invicem ductis, prodit

Aa^m

$Aa^m + a^{\frac{m}{2}} b^{\frac{n}{2}} + ABa^{\frac{m}{2}} b^{\frac{n}{2}} + Bb^n$. Si hoc factum comparetur cum binomio dato $a^m + b^n$, cui ponitur æquale, invenietur $A = 1$, $B = 1$, $ABa^{\frac{m}{2}} b^{\frac{n}{2}} + a^{\frac{m}{2}} b^{\frac{n}{2}} = 0$, adeoque $A = B$, $AA = BB = AB$. Igitur ex $ABa^{\frac{m}{2}} b^{\frac{n}{2}} + a^{\frac{m}{2}} b^{\frac{n}{2}} =$, seu (dividendo per $a^{\frac{m}{2}} b^{\frac{n}{2}}$) ex $AB + 1 = 0$, fit $AA + 1 = 0$, $BB + 1 = 0$, seu $AA = -1$, $BB = -1$, ac proinde $A = \pm\sqrt{-1}$; $B = \pm\sqrt{-1}$, quibus valoribus substitutis, factores $Aa^{\frac{m}{2}} + b^{\frac{n}{2}}$; $a^{\frac{m}{2}} + Bb^{\frac{n}{2}}$ evadent $\pm a^{\frac{m}{2}}\sqrt{-1} + b^{\frac{n}{2}}$; $a^{\frac{m}{2}} \pm b^{\frac{n}{2}}\sqrt{-1}$, ex quibus quatuor eliciuntur factorum paria nimirum

$$\begin{Bmatrix} a^{\frac{m}{2}}\sqrt{-1} + b^{\frac{n}{2}} \\ -a^{\frac{m}{2}}\sqrt{-1} + b^{\frac{n}{2}} \end{Bmatrix} \quad \begin{Bmatrix} a^{\frac{m}{2}} + b^{\frac{n}{2}}\sqrt{-1} \\ a^{\frac{m}{2}} - b^{\frac{n}{2}}\sqrt{-1} \end{Bmatrix}$$

$$\begin{Bmatrix} -a^{\frac{m}{2}}\sqrt{-1} - b^{\frac{n}{2}} \\ a^{\frac{m}{2}}\sqrt{-1} - b^{\frac{n}{2}} \end{Bmatrix} \quad \begin{Bmatrix} -a^{\frac{m}{2}} - b^{\frac{n}{2}}\sqrt{-1} \\ -a^{\frac{m}{2}} + b^{\frac{n}{2}}\sqrt{-1} \end{Bmatrix}$$

profecto ex cujuscumque paris factoribus invicem du-

ductis prodit binomium datum $a^m + b^n$. *Quod erat* &c. Hinc *Regula*. Terminorum alteri, ex quo extracta sit radix quadrata, ducto in $\sqrt{-1}$ præfigatur signum cum positivum, tum negativum; deinde terminus iste tam negativus quam positivus addatur termino altero (ex quo pariter radix quadrata sit elicita) sub signo positivo; ac duo factorum paria prodibunt, quorum singula duos factores quæsitos comprehendent. Quod si mutentur signa, alia duo emergent paria, de quibus sermo est, adeo ut si factores omnes bini sumantur, quatuor factorum paria quæsito satisfacientia reperiantur.

Exemplum. Sit $a = 2$, $m = 1$; erit $a^m = 2$. Sit $b = \sqrt[3]{7}$, $n = 1$; erit $b^n = \sqrt[3]{7}$, quare quantitatis $2 + \sqrt[3]{7}$ factores erunt

$$\left\{\begin{matrix} \sqrt{2}\sqrt{-1} + \sqrt{\sqrt[3]{7}} \\ -\sqrt{2}\sqrt{-1} + \sqrt{\sqrt[3]{7}} \end{matrix}\right\} \qquad \left\{\begin{matrix} \sqrt{2} + \sqrt{\sqrt[3]{7}}\sqrt{-1} \\ \sqrt{2} - \sqrt{\sqrt[3]{7}}\sqrt{-1} \end{matrix}\right\}$$

$$\left\{\begin{matrix} -\sqrt{2}\sqrt{-1} - \sqrt{\sqrt[3]{7}} \\ \sqrt{2}\sqrt{-1} - \sqrt{\sqrt[3]{7}} \end{matrix}\right\} \qquad \left\{\begin{matrix} -\sqrt{2} - \sqrt{\sqrt[3]{7}}\sqrt{-1} \\ -\sqrt{2} + \sqrt{\sqrt[3]{7}}\sqrt{-1} \end{matrix}\right\}$$

537. *Coroll.* 1. Ergo quilibet numerus, etiam primus, plura pro sua magnitudine habebit factorum imaginariorum paria. Ex. gr. si numerus 3 in omnes summas ipsum per duos numeros efficientes dividatur, quatuor per hanc methodum factorum paria habebit quoad $1+2$, & quatuor quoad $2+1$. Generatim si numerus quilibet vocetur n, ille in omnes binos numeros ipsum efformantes divisus tot factorum paria possidebit, quot indicat numerus $4(n-1)$.

538. *Coroll.* 2. Si fuerit trinomium $a^m + b^n + c^r$, cujus varia quærantur factorum paria; reducatur ad binomium, statuendo $b^n + c^r = d^r$. Binomii igitur $a^m + d^r$ factoribus secundum Regulam inventis, si fiant substitutiones, habebis

$$\left\{\begin{array}{c} \sqrt{a^m}\sqrt{-1}+\sqrt{(b^n+c^r)} \\ -\sqrt{a^m}\sqrt{-1}+\sqrt{(b^n+c^r)} \end{array}\right\}$$

$$\left\{\begin{array}{c} \sqrt{(b^n+c^r)}\sqrt{-1}+\sqrt{a^m} \\ -\sqrt{(b^n+c^r)}\sqrt{-1}+\sqrt{a^m} \end{array}\right\}$$

quatuor nempe ut supra factorum paria si mutentur signa. Sed trinomium $a^m + b^n + c^r$ tribus modis

Pars alt. K ad

ad binomium redigi poteſt, hoc eſt faciendo $a^m +$ $(b^n + c^r)$; $b^n + (a^m + c^r)$; $c^r + (a^m + b^n)$; ergo inter trinomii propoſiti factores paria duodecim invenientur. Pari modo procedendo, ſi quadrinomii $a^m + b^n + c^r + d^s$ factores ut ante quærantur, reperientur viginti & octo paria, quadrinomio ad binomium per omnes recenſitas ſubſtitutiones reducto; & ſic de ceteris. Ex quo colligitur, polynomii cujuſlibet plura factorum paria obtineri poſſe, etſi quivis polynomii terminus quantitatem etiam radicalem continens ad poteſtates diverſas aſſurgat.

539. *Coroll.* 3. Ut inveniantur quatuor factorum paria binomii $a^m - b^n$, ſimili ratiocinio procedendum. Etenim finge, iſtos eſſe $Aa^{\frac{m}{2}} + b^{\frac{n}{2}}$, $a^{\frac{m}{2}} - Bb^{\frac{n}{2}}$. Peractis operationibus, ut Problematis reſolutio docet, duo tantum factorum paria habebis, videlicet

$$\left\{\begin{matrix} a^{\frac{m}{2}} - b^{\frac{n}{2}} \\ a^{\frac{m}{2}} + b^{\frac{n}{2}} \end{matrix}\right\} \qquad \left\{\begin{matrix} -a^{\frac{m}{2}} + b^{\frac{n}{2}} \\ -a^{\frac{m}{2}} - b^{\frac{n}{2}} \end{matrix}\right\}$$

Ut alia duo paria obtineas, pone, factorum alterum eſſe $Aa^{\frac{m}{2}} + b^{\frac{n}{2}}\sqrt{-1}$, alterum $a^{\frac{m}{2}}\sqrt{-1} - Bb$

$Bb^{\frac{n}{2}}$, quibus invicem multiplicatis, prodit $Aa^{m}\sqrt{-1} - a^{\frac{m}{2}}b^{\frac{n}{2}} - ABa^{\frac{m}{2}}b^{\frac{n}{2}} - Bb^{n}\sqrt{-1}$; & instituta comparatione, reperies $A\sqrt{-1} = 1$, $B\sqrt{-1} = 1$; quare $A = B = \frac{1}{\sqrt{-1}} = -\sqrt{-1}$, adeoque $AB = AA = BB = -1$. Hinc $\sqrt{AB} = A = B = \pm\sqrt{-1}$, quo valore in factoribus indeterminatis $Aa^{\frac{m}{2}} + b^{\frac{n}{2}}\sqrt{-1}$, $a^{\frac{m}{2}}\sqrt{-1} - Bb^{\frac{n}{2}}$ substituto, provenit $\pm a^{\frac{m}{2}}\sqrt{-1} + b^{\frac{n}{2}}\sqrt{-1}$; $a^{\frac{m}{2}}\sqrt{-1} \mp b^{\frac{n}{2}}\sqrt{-1}$; unde duo factorum exurgunt paria

$$\left\{\begin{matrix} a^{\frac{m}{2}}\sqrt{-1} + b^{\frac{n}{2}}\sqrt{-1} \\ -a^{\frac{m}{2}}\sqrt{-1} + b^{\frac{n}{2}}\sqrt{-1} \end{matrix}\right\}$$

$$\left\{\begin{matrix} -a^{\frac{m}{2}}\sqrt{-1} - b^{\frac{n}{2}}\sqrt{-1} \\ a^{\frac{m}{2}}\sqrt{-1} - b^{\frac{n}{2}}\sqrt{-1} \end{matrix}\right\}$$

ex quo liquet, residuum quoque $a^{m} - b^{n}$ quatuor habere posse factorum paria.

540. *Coroll.* 4. Itaque cujuslibet polynomii, cujus termini sint signis diversis connexi, plura fa-

ctorum paria possunt obtineri. Detur enim $a^m - b^n + c^r$; si fiat $a^m - b^n = d^t$, habebimus $c^r + d^t$, cujus factores ut supra per Regulam erunt in promptu. At si statuamus $b^n + c^r = g^v$, orietur residuum $a^m - g^v$, cujus enunciatos factores per Coroll. anteced. obtinebimus. Idem eveniet, si trinomium fuerit $a^m - b^n - c^r$; & sic de reliquis hujusmodi polynomiis.

541. *Coroll.* 5. Si ponatur $a = b$, $m = n$, primum & quartum binomii $a^m + b^m$ factorum par evadet $\left\{\begin{array}{l} a^{\frac{m}{2}}\sqrt{-1} + a^{\frac{m}{2}} \\ -a^{\frac{m}{2}}\sqrt{-1} + a^{\frac{m}{2}} \end{array}\right\}$

$\left\{\begin{array}{l} -a^{\frac{m}{2}} - a^{\frac{m}{2}}\sqrt{-1} \\ -a^{\frac{m}{2}} + a^{\frac{m}{2}}\sqrt{-1} \end{array}\right\}$; sed singula ista paria invicem multiplicando prodit $2a^m$; ergo si statuatur $a^m = \frac{1}{2}f$, & valor iste substituatur, duo potestatis

f^m

f^m paria factorum exurgent, videlicet

$$\left\{\begin{array}{l} \sqrt{\tfrac{1}{2}f^m}\sqrt{-1}+\sqrt{\tfrac{1}{2}f^m} \\ -\sqrt{\tfrac{1}{2}f^m}\sqrt{-1}+\sqrt{\tfrac{1}{2}f^m} \end{array}\right\}$$

$$\left\{\begin{array}{l} -\sqrt{\tfrac{1}{2}f^m}-\sqrt{\tfrac{1}{2}f^m}\sqrt{-1} \\ -\sqrt{\tfrac{1}{2}f^m}+\sqrt{\tfrac{1}{2}f^m}\sqrt{-1} \end{array}\right\}$$

est enim $a^{\frac{m}{2}}=\sqrt{a^m}=\sqrt{\frac{1}{2}f^m}$. Substitutiones in reliquis duobus factorum paribus eodem recidunt; quapropter hæc methodus non exhibet nisi duo factorum paria. Verum alia duo $f^{\frac{m}{2}}$, $f^{\frac{m}{2}}$; $-f^{\frac{m}{2}}$, $-f^{\frac{m}{2}}$ sunt per se manifesta; quare quatuor hic etiam factorum paria quantitatis f^m nanciscimur. Sed alia duo, quin etiam infinita, levi adhibita attentione possumus expiscari; quum enim duo factores $\frac{1}{2}a^{\frac{m}{2}}\sqrt{-1}$

$+\frac{1}{2}a^{\frac{m}{2}}, -\frac{1}{2}a^{\frac{m}{2}}\sqrt{-1}+\frac{1}{2}a^{\frac{m}{2}}$ inter se ducti efficiant $\frac{1}{2}a^{m}$, statim dignoscitur, quod si termini, ubi adsunt radices imaginariæ, ita immutentur, ut quantitatis $\frac{1}{2}a^{m}$ numerus $\frac{1}{2}$ eliminetur, nova prodibunt factorum paria. In locum igitur $\sqrt{-1}$ sufficio $\sqrt{-3}$, & pro duobus paribus factorum quantitatis a^{m} habeo

$$\left\{\begin{array}{l} \frac{1}{2}a^{\frac{m}{2}}\sqrt{-3}+\frac{1}{2}a^{\frac{m}{2}} \\ -\frac{1}{2}a^{\frac{m}{2}}\sqrt{-3}+\frac{1}{2}a^{\frac{m}{2}} \end{array}\right\}$$

$$\left\{\begin{array}{l} -\frac{1}{2}a^{\frac{m}{2}}\sqrt{-3}-\frac{1}{2}a^{\frac{m}{2}} \\ \frac{1}{2}a^{\frac{m}{2}}\sqrt{-3}-\frac{1}{2}a^{\frac{m}{2}} \end{array}\right\}$$

Innumera vero sic exhibeo

$\frac{1}{p}$

$$\left\{\begin{matrix} \frac{1}{r} a^{\frac{m}{2}} \sqrt{-(r^2-1)} + \frac{1}{r} a^{\frac{m}{2}} \\ -\frac{1}{r} a^{\frac{m}{2}} \sqrt{-(r^2-1)} + \frac{1}{r} a^{\frac{m}{2}} \end{matrix}\right\}$$

$$\left\{\begin{matrix} -\frac{1}{r} a^{m-1} \sqrt{-(r^2-1)} - \frac{1}{r} a^{m-1} \\ \frac{1}{r} a \sqrt{-(r^2-1)} - \frac{1}{r} a \end{matrix}\right\}$$

Vel generalius

$$\left\{\begin{matrix} \frac{-1}{r^{\frac{m}{2}}} a^{m-1} \sqrt{-(r^m-1)} - \frac{1}{r^{\frac{m}{2}}} a^{m-1} \\ \frac{1}{r^{\frac{m}{2}}} a \sqrt{-(r^m-1)} - \frac{1}{r^{\frac{m}{2}}} a \end{matrix}\right\}$$

$$\left\{\begin{matrix} \frac{1}{r^{\frac{m}{2}}} a^{\frac{m}{2}} \sqrt{-(r^m-1)} + \frac{1}{r^{\frac{m}{2}}} a^{\frac{m}{2}} \\ -\frac{1}{r^{\frac{m}{2}}} a^{\frac{m}{2}} \sqrt{-(r^m-1)} + \frac{1}{r^{\frac{m}{2}}} a^{\frac{m}{2}} \end{matrix}\right\}$$

CAPUT VI.

DE RADICUM EX BINOMIIS EXTRACTIONE, UBI DE RADICE QUADRATA TAM EX MONOMIO IMAGINARIO, QUAM EX QUOCUMQUE POLYNOMIO ELICIENDA.

542. Propositio I. *Ex monomio imaginario in* $\sqrt{-1}$ *radicem quadratam extrahere.*

Resolutio. Fiat $\sqrt{m\sqrt{-1}} = x + y\sqrt{-1}$. Quadrando habetur $m\sqrt{-1} = x^2 - y^2 + 2xy\sqrt{-1}$. Itaque si comparentur termini rationales, vel si memoriæ inhæreant quæ sunt alibi demonstrata (437.), stabit $x^2 - y^2 = 0$, adeoque $x = y$, & $2xy = m$. Hinc $xy = yy = \frac{m}{2}$, & $y = \pm\sqrt{\frac{m}{2}}$. Ergo generatim $\sqrt{m\sqrt{-1}} = x + y\sqrt{-1} = y + y\sqrt{-1} = \pm\sqrt{\frac{m}{2}}(1 + \sqrt{-1})$. *Quod erat* &c.

Exemplum I. Elicienda esto radix ex $2\sqrt{-1}$. Erit $m = 2$; quare $\sqrt{2\sqrt{-1}} = \pm\sqrt{\frac{2}{2}}(1 + \sqrt{-1}) = 1 + \sqrt{-1}$, vel $= -1 - \sqrt{-1}$.

Exem-

Exemplum II. Sit extrahenda radix ex $8\sqrt{-1}$. Erit $m=8$; hinc $\sqrt{8\sqrt{-1}} = \pm\sqrt[8]{1}(1+\sqrt{-1}) = 2+2\sqrt{-1}$, vel $= -2-2\sqrt{-1}$.

543. Propositio II. *Ex binomio radicem quadratam extrahere.*

Resolutio. Binomium, ex quo radix quadrata proponitur elicienda, repræsentetur per $A \pm B$, ubi monendum in casu differentiæ $A-B$, esse debere $A>B$, secus binomium $A-B$ quadrati negativi formam indueret, adeoque radicis extractio ex eo foret impossibilis. Finge interea, esse $x+y=\pm\sqrt{(A+B)}$, atque $x-y=\pm\sqrt{(A-B)}$. Addendo erit $x = \frac{\pm\sqrt{(A+B)}\pm\sqrt{(A-B)}}{2}$; quadrando $xx = \frac{A+\sqrt{(A^2-B^2)}}{2}$; & radicem extrahendo, $x = \pm\sqrt{\frac{A+\sqrt{(A^2-B^2)}}{2}}$. Pari methodo, sed primum subtrahendo, invenitur $y = \pm\sqrt{\frac{A-\sqrt{(A^2-B^2)}}{2}}$; quare $x+y = \pm\sqrt{\frac{A+\sqrt{(A^2-B^2)}}{2}} \pm$

$\sqrt{}$

$$\sqrt{\frac{A-\sqrt{(A^2-B^2)}}{2}};\ x-y=\pm\sqrt{\frac{A+\sqrt{(A^2-B^2)}}{2}}$$

$$\mp\sqrt{\frac{A-\sqrt{(A^2-B^2)}}{2}}$$; ex quo (ſignorum combinationibus utrinque ſeparatis) fit generatim $x\pm y$

$$=\sqrt{\frac{A+\sqrt{(A^2-B^2)}}{2}}\pm\sqrt{\frac{A-\sqrt{(A^2-B^2)}}{2}};$$

ſive $x\pm y=-\sqrt{\frac{A+\sqrt{(A^2-B^2)}}{2}}\mp\sqrt{\frac{A-\sqrt{(A^2-B^2)}}{2}}$.

Quum itaque $\sqrt{(A^2-B^2)}$ eſſe debeat (*ex hyp.*) quantitas realis, radix binomii propoſiti per binomium radicale ſemper exprimetur quotieſcumque ex A^2-B^2 radix extrahi poterit; *Quod erat* &c.

Exemplum I. Sit binomium $5\pm\sqrt{24}$, cujus quæratur radix quadrata. Erit $a=5$; $\pm B=\pm\sqrt{24}$;

$$A^2-B^2=25-24=1;\ x\pm y=\sqrt{\frac{5+1}{2}}\pm$$

$$\sqrt{\frac{5-1}{2}}=\sqrt{3}\pm\sqrt{2}$$; & radices omnes affirmativas, & negativas ſingillatim exhibendo, radices iſtæ erunt quatuor, nimirum $\sqrt{3}+\sqrt{2}$; $-\sqrt{3}-\sqrt{2}$; $\sqrt{3}-\sqrt{2}$, $\sqrt{2}-\sqrt{3}$; priores ſcilicet duæ pro binomio $5+\sqrt{24}$, reliquæ pro binomio $5-\sqrt{24}$.

Exem-

Exemplum II. Binomium datum esto $\sqrt{32} \pm \sqrt{24}$. Ergo $A = \sqrt{32}$; $\pm B = \pm\sqrt{24}$; $A^2 - B^2 = 8$;

$$\frac{A + \sqrt{(A^2 - B^2)}}{2} = \frac{\sqrt{32} + \sqrt{8}}{2} = \frac{4\sqrt{2} + 2\sqrt{2}}{2} =$$

$$3\sqrt{2};\ \frac{A - \sqrt{(A^2 - B^2)}}{2} = \frac{\sqrt{32} - \sqrt{8}}{2} = \sqrt{2};\ x \pm y$$

$$= \sqrt{3\sqrt{2}} \pm \sqrt{\sqrt{2}} = \sqrt{\sqrt{18}} \pm \sqrt{\sqrt{2}} = \sqrt[4]{18} \pm \sqrt[4]{2}.$$

Exemplum III. Extrahi debeat radix ex binomio $z^2 + 2c\sqrt{(z^2 - c^2)}$. Ergo $A = z^2$; $\pm B = \pm 2c\sqrt{(z^2 - c^2)}$; $\sqrt{(A^2 - B^2)} = \sqrt{(z^4 - 4z^2c^2 + 4c^4)} = z^2 - 2c^2$; quare

$$x \pm y = \sqrt{\frac{(z^2 + z^2 - 2c^2)}{2}} \pm \sqrt{\frac{(z^2 - z^2 + 2c^2)}{2}} =$$

$\sqrt{(z^2 - c^2)} \pm c$. Quatuor igitur erunt radices, videlicet $\sqrt{(z^2 - c^2)} + c$, $-\sqrt{(z^2 - c^2)} - c$; $\sqrt{(z^2 - c^2)} - c$, $-\sqrt{(z^2 - c^2)} + c$; primæ duæ pro binomii aggregato, reliquæ duæ pro ejusdem differentia.

Exemplum IV. Postuletur radix binomii $-5 \pm \sqrt{24}$. Ergo $A = -5$; $\pm B = \pm\sqrt{24}$; $A^2 - B^2 = 25 -$

$$24 = 1;\ x \pm y = \sqrt{\frac{-5 + 1}{2}} \pm \sqrt{\frac{-5 - 1}{2}} =$$

$\sqrt{-2} \pm \sqrt{-3}$. Quapropter radices erunt quatuor, nempe $\sqrt{-2} + \sqrt{-3}$, $-\sqrt{-2} - \sqrt{-3}$; $\sqrt{-2} - \sqrt{-3}$, $\sqrt{-3} - \sqrt{-2}$; duæ priores ad binomium $-5 + \sqrt{24}$, reliquæ ad binomium $-5 - \sqrt{24}$ pertinentes.

Exem-

Exemplum V. Educenda sit radix ex binomio $-7 \pm 4\sqrt{-2}$. Ergo $A = -7$; $\pm B = \pm 4\sqrt{-2}$; $A^2 - B^2 = 81$; $x \pm y = 1 \pm 2\sqrt{-2}$. Rursus tento, an ex binomio $1 \pm 2\sqrt{-2}$ radicem extrahere possim; quare $A = 1$; $\pm B = \pm 2\sqrt{-2}$; $A^2 - B^2 = 9$. Hinc $x \pm y = \sqrt{2} \pm \sqrt{-1}$. Ex quo constat, allatam formulam extractioni quoque altiorum radicum, quarum exponens est numerus par, inservire; sufficit enim ad hujusmodi finem adsequendum, ut eadem operatio toties iteretur, quoties numerus 2 dictum radicum exponentem ingreditur. Eadem formula radicibus ex trinomiis eruendis est utilis, ut docet sequens.

Exemplum VI. Esto ex trinomio $m\sqrt{t} \pm 2\sqrt{mn\sqrt{tv}} + n\sqrt{v}$ radix extrahenda. Ergo $A = m\sqrt{t} + n\sqrt{v}$;

$\pm B = \pm 2\sqrt{mn\sqrt{tv}}$; $A^2 - B^2 = m^2t + n^2v - 2mn\sqrt{tv}$; $\sqrt{(A^2 - B^2)} = m\sqrt{t} - n\sqrt{v}$; quare $x \pm y$

$= \sqrt{m\sqrt{t}} \pm \sqrt{n\sqrt{v}}$; quatuor igitur erunt radices,

nimirum $\sqrt{m\sqrt{t}} + \sqrt{n\sqrt{v}}$, $-\sqrt{m\sqrt{t}} - \sqrt{n\sqrt{v}}$;

$\sqrt{m\sqrt{t}} - \sqrt{n\sqrt{v}}$, $\sqrt{n\sqrt{v}} - \sqrt{m\sqrt{t}}$; quarum duæ

priores trinomium $m\sqrt{t} + 2\sqrt{mn\sqrt{tv}} + n\sqrt{v}$, duæ

posteriores trinomium $m\sqrt{t} - 2\sqrt{mn\sqrt{tv}} + n\sqrt{v}$ respiciunt.

At

At si eandem methodum in quadrinomiis, ceterisque polynomiis experiri lubeat, res ita evadet difficultatibus implexa, ut ad Regulam Newtonianam satius sit confugere; quam ut e suo fonte derivemus, sequens est præmittendum.

544. Lemma. *Sint* a, b *duæ quantitates incommensurabiles ejus conditionis, ut earum quadrata* aa, bb *sint utraque rationalia; dico, earundem factum esse irrationale.*

Dem. Si enim *a* sit quantitati *b* incommensurabilis, idem de earum multiplis, ac submultiplis, continget; ergo etiam *ab* erit quantitati *bb* incommensurabilis; sed *bb* est (*per hyp.*) quantitas rationalis; ergo *ab* est ipsi *bb* irrationalis. *Quod erat* &c.

545. Propositio III. *Ex polynomio quolibet, in quo* (1.°) *singulorum terminorum quadrata sint rationalia, dum omnes termini, vel omnes, uno excepto, sint irrationales,* (2.°) *termini sint vicissim incommensurabiles, nec ad minorem numerum reducibiles; radicem quadratam extrahere.*

Resolutio. Sit quadrinomium $v-u+y-z$, quo evecto ad quadratum $v^2+u^2+y^2+z^2-2vu+2vy-2vz-2uy+2uz-2yz$, sequentia sunt prænoscenda; (1.°) partes v^2, u^2, y^2, z^2 numerus componunt rationales, dum reliquæ $2vu$, $2vy$, $2vz$, $2uy$, $2uz$, $2yz$ vi lemmatis sunt omnes irrationales; (2.°) partes singulæ irrationales duos radicis terminos continent, quare duæ istarum partium ita possunt invicem multiplicari, ut producto per aliquam ex remanentibus diviso, quotus emergat numerum

re-

referens rationalem; sic $\frac{2vx \times 2vy}{2xy} = 2v^2$, pariterque $\frac{2xy \times 2yz}{2xz} = 2y^2$. Hinc (3.°) si duæ partes irrationales invicem ductæ per reliquarum aliquam dividantur, ita ut quotus quantitatem rationalem exhibeat, & ex hujus quoti dimidio radix extrahatur, ista erit unus ex terminis radicis quadratum, quod examini subjecimus, efficientis. Sic dimidium quoti $2v^2$ est quantitas rationalis v^2, cujus radix v est unus ex terminis radicem quadrati recensiti componentibus. Idem de quoto $2y^2$. Sed ista in hujusmodi polynomiis omnibus reperiuntur; ergo generatim

I. Duæ polynomii partes irrationales invicem ductæ per reliquarum aliquam dividantur, ita ut quotus rationalis emergat; ex hujus dimidio extrahatur radix; hæc erit unus ex terminis polynomii radicem ingredientibus.

II. Quoad signa terminis jam inventis præfigenda, advertendum est, partium irrationalium signo positivo affectarum factores ex signis similibus coalescere, signo negativo præditarum factores ex signis dissimilibus conflari; pro quibus dignoscendis numerus 2 tantummodo debet extra quantitates irrationales semper remanere. Ut autem certius procedatur, sumantur primum quantitates illæ, quæ signis similibus gaudent & ponantur seorsum cum signis positivis in prima columna, cum negativis in

in fecunda; tum fi duæ aliæ hujufmodi quantitates, quarum una fit ex jam feparatis, figna habeant diverfa, & uni ex iifdem datum antea fit fignum pofitivum, altera cum figno negativo collocetur in prima columna, & contra. Cautionibus hifce rite adhibitis, radicem ex recenfitis polynomiis, dummodo quadratum perfectum referant, eliiciemus. *Quod erat* &c.

Exemplum I. Radix ex quadrinomio $14-\sqrt{72}+\sqrt{108}-\sqrt{24}$ elicienda efto. Ut quadrinomium ad normam allatæ methodi reducam, divido per 4 quantitates fub figno radicali exiftentes, ac fub eodem figno relicto quoto, numerum 2 extra fignum radicale affigo; Quadrinomium igitur datum fic reductum erit $14-2\sqrt{18}+2\sqrt{27}-2\sqrt{6}$. Sumo interea $\frac{2\sqrt{18}\times2\sqrt{6}}{2\sqrt{27}}=2\sqrt{4}=4$, quæ eft quantitas rationalis; ex hujus dimidio 2 radicem extraho, nimirum $\sqrt{2}$, quæ eft unus ex radicis quæfitæ terminis; deinde feligo $\frac{2\sqrt{18}\times2\sqrt{27}}{2\sqrt{6}}=18$, cujus dimidium 9, ejufque radix 3, quæ eft alter radicis terminus. Supereft $\frac{2\sqrt{27}\times2\sqrt{6}}{2\sqrt{18}}=6$, cujus dimidium 3, ejufque radix $\sqrt{3}$, quæ eft tertius radicis terminus. Termini igitur radicis quæfitæ funt $3, \sqrt{2}, \sqrt{3}$.

Per-

Pergo ad signa. Ut ea distribuam, quoniam $\sqrt{27}$ constat ex factoribus 3, & $\sqrt{3}$, infero, duos hosce factores signis similibus affici; quare $+3$, & $+\sqrt{3}$ scribo in prima columna *A*, pariterque -3, & $-\sqrt{3}$ colloco in secunda columna *B*. Tum ex $-\sqrt{6}$, quæ ob signum negativum oritur ex factoribus $\sqrt{2}$, & $\sqrt{3}$ signa dissimilia habentibus, coniicio, quod si statuatur $\sqrt{3}$ factor positivus, esse debet $\sqrt{2}$ negativus, & contra; adeoque in columna *A* ubi adest $+\sqrt{3}$, pono $-\sqrt{2}$, & contra in columna *B*. Itaque concludo, radicem quæsitam esse vel $3-\sqrt{2}+\sqrt{3}$, vel $-3+\sqrt{2}-\sqrt{3}$, ut examen instituenti palam erit.

A	*B*
$+3$	-3
$+\sqrt{3}$	$-\sqrt{3}$
$-\sqrt{2}$	$+\sqrt{2}$

Exemplum II. Proponatur eruenda radix ex polynomio reducto $16+2\sqrt{6}-2\sqrt{10}-2\sqrt{12}-2\sqrt{15}-2\sqrt{18}+2\sqrt{30}$. Sumo primū $\frac{2\sqrt{6}\times 2\sqrt{10}}{2\sqrt{12}}=2\sqrt{5}$; sed quotus iste non est rationalis, ergo hoc rejecto, transeo ad $\frac{2\sqrt{6}\times 2\sqrt{10}}{2\sqrt{15}}=4$, cujus dimidium 2, ejusque radix $\sqrt{2}$ est unus ex terminis radicis quæsitæ; eadem adhibita cautione invenio $\frac{2\sqrt{6}\times 12\sqrt{18}}{2\sqrt{12}}=6$, adeoque $\sqrt{3}$ alter radicis terminus. Habeo præterea $\frac{2\sqrt{18}\times 2\sqrt{30}}{2\sqrt{15}}=12$, quare $\sqrt{6}$ tertium radi-

radicis terminum constituit. Tandem $\frac{2\sqrt{15}\times 2\sqrt{30}}{2\sqrt{18}}$ $= 10$ præbet $\sqrt{5}$ pro quarto radicis termino. Quatuor igitur radicis termini sunt $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}$. Sed ob $+\sqrt{6}$ duo ejus factores $\sqrt{2}$, & $\sqrt{3}$, signis consimilibus gaudent, & ob $-\sqrt{12}$, pariterque ob $-\sqrt{15}$, signa sunt dissimilia in factoribus $\sqrt{2}$, & $\sqrt{6}$, nec non in $\sqrt{3}$, & $\sqrt{5}$; ergo pro radice imperata stabit $\sqrt{2}+\sqrt{3}-\sqrt{5}-\sqrt{6}$, sive $\sqrt{5}+\sqrt{6}-\sqrt{3}-\sqrt{2}$.

Exemplum III. Elici jubeatur radix ex polynomio jam reducto $-5-2\sqrt{-6}-2\sqrt{2}-2\sqrt{-10}+2\sqrt{-3}-2\sqrt{15}+2\sqrt{-5}$. Quoniam in radicibus imaginariis dari possunt sub signo radicali quantitates affirmativæ, quæ a negativis invicem ductis componantur (ex. gr. $2\sqrt{-3}-2\sqrt{15}$ prodire potest tam ex $2\sqrt{-3}-2\sqrt{3}\times\sqrt{5}$, quam ex $2\sqrt{-3}+\sqrt{-3}\times\sqrt{-5}$), ad errorem vitandum dum radicis quæsitæ terminos venamur, retineantur signa radicalibus illis præfixa, ex quibus invicem ductis, ac deinde ut diximus divisis, quotus expetitus emergere debet.

Sic $\frac{-\sqrt{-6}\times-\sqrt{2}}{\sqrt{-3}} = 2$, cujus radix $\sqrt{2}$ est unus ex terminis radicis quæsitæ. Itidem $\frac{-\sqrt{15}\times\sqrt{-5}}{\sqrt{-3}}$ $= -\sqrt{25} = \sqrt{-5}\times\sqrt{-5} = -5$, cujus radix $\sqrt{-5}$ est alter ex terminis ejusdem radicis. Præ-

terea $\frac{\sqrt{-3}\times-\sqrt{15}}{\sqrt{-5}} = -3$, cujus radix $\sqrt{-3}$

eſt tertius radicis terminus. Denique $\frac{\sqrt{-3}\times\sqrt{-5}}{-\sqrt{15}}$

$=1$, cujus radix 1 eſt quartus radicis terminus. Juſtæ igitur radicis termini erunt $1, \sqrt{2}, \sqrt{-3}$, $\sqrt{-5}$. Quoad ſigna, quantitates 1, & $-\sqrt{2}$ habent ſigna diſſimilia; ſed 1, & $\sqrt{-3}$, pariterque 1, & $\sqrt{-5}$ ſigna habent ſimilia; ergo radix quæſita erit $1-\sqrt{2}+\sqrt{-3}+\sqrt{-5}$, ſive $\sqrt{2}-1-\sqrt{-3}-\sqrt{-5}$; quod examine comprobatur.

546. Lemma. *Binomii, cujus uterque terminus ad quadratum elatus ſit rationalis, quænam ſint radices cubicæ fractionariæ, determinare.*

Reſolutio. Radices iſtæ exprimantur per $\frac{x}{m}+\frac{y}{n}$, in quibus x, m ſtatuantur numeri inter ſe primi, & y radicem numeri integri repræſentans nullam cum numero n reductionem patiatur. Evehatur $\frac{x}{m}+\frac{y}{n}$ ad cubum. Pars rationalis erit $\frac{x^3}{m^3}+\frac{3xy^2}{mn^2}$; irrationalis $\left(\frac{3x^2}{nm^2}+\frac{y^2}{n^3}\right)y$. Quoniam vero (*ex hyp.*) tam $\frac{x^3}{m^3}+\frac{3xy^2}{mn^2}$, quam $\frac{3x^2}{nm^2}+\frac{y^2}{n^3}$ eſſe de-

bent

bent numeri integri, ponatur $\frac{3x^2}{m^2n} + \frac{y^2}{n^3} = g$. Ergo $y^2 = gn^3 - \frac{3x^2n^2}{m^2}$; quum autem hæc quantitas esse debeat numerus integer, erit quoque talis $\frac{3x^2n^2}{m^2}$, ac proinde n^2 multiplus quantitatis m^2. Posito igitur $n = mb$, stabit $y^2 = gm^3b^3 - 3x^2b^2$, quibus valoribus in formula $\frac{x^3}{m^3} + \frac{3xy^2}{mn^2}$ substitutis, oritur $-\frac{8x^3}{m^3} + 3xgb$, qui numerus integer esse debet. Sed x, & m nullum habent [*ex hyp.*] communem divisorem; ergo m esse nequit nisi 1, vel 2. Præterea quoniam $y^2 = gm^3b^3 - 3x^2b^2$ exhibet $\frac{y}{b} = \sqrt{gbm^3 - 3x^2}$, adeoque $\frac{y}{mb} = \frac{y}{n} = \frac{1}{m}\sqrt{gbm^3 - 3x^2}$, sequitur, alteram radicis cubicæ partem eodem cum prima denominatore 1, vel 2 præditam esse. *Quod erat* &c.

547. Propositio IV. *Ex binomio, cujus uterque terminus ad quadratum evectus sit rationalis, radicem cubicam extrahere.*

Resolutio. Binomium partim rationale, partim irrationale ad simpliciorem expressionem redactum sit $a + b\sqrt{c}$. Quoniam istud veluti cubus considera-

 tur,

tur, liquet, in ejus radice, qualiscumque ea sit, includi debere quantitatem irrationalem $\sqrt{c}$, idque ex cognita cuborum proprietate, nisi enim $\sqrt{c}$ in radice cubica adesset, neque in cubo reperiretur, ex. gr. cubus $2\sqrt{2}$ habet $\sqrt{2}$ pro radice, quæ proinde tam in cubo quam in radice necessario apparet, & sic de reliquis. Quum itaque cubus $a + b\sqrt{c}$ terminis altero rationali, altero irrationali ex hypothesi constet, etiam radix terminis altero rationali, irrationali altero constare debet.

Hisce præmonitis, radix cubica elicienda ponatur jam inventa, nempe $z + m\sqrt{y}$, in qua z quantitatem rationalem, $\sqrt{y}$ irrationalem, & m hujus multiplum designent. Radicis cubus erit $z^3 + 3z^2m\sqrt{y} + 3m^2zy + m^3y\sqrt{y}$, cujus pars rationalis $z^3 + 3m^2zy$, irrationalis $3z^2m\sqrt{y} + m^3y\sqrt{y}$. Sed $\sqrt{y} = \sqrt{c}$, & $y = c$; ergo $3z^2m\sqrt{y} + m^3y\sqrt{y} = 3z^2m\sqrt{c} + m^3c\sqrt{c} = b\sqrt{c}$, quare $3z^2m + m^3c = b$, seu $z^2 = \frac{b - m^3c}{3m}$, & $z = \pm\sqrt{\left(\frac{b - m^3c}{3m}\right)}$.

Ut autem z in terminis penitus cognitis haberi possit, ponatur m æqualis numero aliquo, adhibita tamen cautione ne radix evadat imaginaria; tum valores incognitarum z, m substituantur in parte rationali nuper separata $z^3 + 3m^2zc$. Quod si quantitas inde prodiens non evadat æqualis termino rationali a dati binomii $a + b\sqrt{c}$, alia quantitas pro incognita m adsumatur, donec hujusmodi æqualitas, si fieri potest, obtineatur; tunc vero radix cubica quæsita erit inventa. *Quod erat* &c.

548.

548. *Monita.* Antequam ad Exempla descendamus, præmonendum.

(1.°) Quum quantitas x substituenda sit in formula $x^3 + 3m^2xc$, quæ debet esse rationalis, liquet, quantitatem $\frac{b - m^3c}{3m}$ esse debere quadratum perfectum, adeoque substitutio pro m (in quo tentando ab unitate incipi debet) facilior redditur.

(2.°) Binomii terminos eodem signo positivo semper præditos accipere possumus; inventa enim radice cubica $x + y = \sqrt[3]{(a + b\sqrt{c})}$, statim habetur radix cubica $x - y = \sqrt[3]{(a - b\sqrt{c})}$, cubi enim hujusmodi non nisi per signa alterna inter se differunt.

(3.°) Si cubus propositus habeat utrosque terminos negativos; sumantur isti affirmativi, & ex binomio affirmativo extracta radice cubica, in hac signa mutentur.

(4.°) Si binomium datum habeat divisores, seu fractiones; reducantur istæ prius ad eundem denominatorem, tum radix cubica numeratoris inventa dividatur per radicem cubicam denominatoris.

(5.°) Si binomii propositi uterque terminus radicalibus quadraticis afficiatur; multiplicetur per cubum radicalis illius, quo terminus alteruter afficitur, & unus ex terminis fiet rationalis, dum alter irrationalis permanebit; tum radix cubica elicita dividatur per radicalem cubi, per quem binomium fuit multiplicatum. Sic $a\sqrt{f} + b\sqrt{g}$ si ducatur in $f\sqrt{f}$,

fit $aff + bf\sqrt{fg}$, ubi quum terminus aff fit rationalis, casus iste secundus ad primum reducitur.

(6.°) Oportet interdum dati binomii terminum radicali affectum transformare, ut radix cubica, quæ prius inveniri non poterat, sic agendo, si adest, inveniatur.

(7.°) Si quantitas extra signum radicale non sit quantum expedit major quantitate illa, quæ sub eodem signo latet, methodus irrita evadit, ut apertum est. Tunc autem per aliam quantitatem binomium datum multiplicare oportet, ita ut productum, si fieri potest, evadat cubus, ex quo post extractam radicem, ista per radicem cubicam quantitatis illius, per quam facta fuit multiplicatio, dividatur.

(8.°) Si binomii carentis fractione in suis terminis, radicem allatis diligentiis adhibitis inquirens, eam adsequi nunquam potest, multiplicet vi lemmatis (546.) per 8 dictum binomium, & extractam radicem cubicam dividat per 2. Quod si nec ita operando voti fiet compos, certus erit, binomium propositum non esse cubum perfectum.

Hæc omnia fient Exemplis illustria. Interea ad memoriam adjuvandam sit in prospectu sequens Tabella

Binomium cubicum	*Pars cubi rationalis*	*Radix cubica*
$a + b\sqrt{c}$	$z^3 + 3m^2cz$	$z + m\sqrt{c}$

Exem-

Exempla.

I. Sit extrahenda radix cubica ex binomio $26 + \sqrt{675}$. Hoc ad ſimplicem formam redactum evadit $26 + 15\sqrt{3}$. Ergo $b = 15$, $c = 3$. Hinc $\pm \sqrt[3]{\left(\frac{b - m^3 c}{3m}\right)}$ $= \pm \sqrt[3]{\left(\frac{15 - 3m^3}{3m}\right)} = \pm \sqrt[3]{\left(\frac{5 - m^3}{m}\right)}$; ergo ut hæc quantitas fiat rationalis, eſſe debet $m = 1$, adeoque $\pm 2 = z$. Suffice hos valores omnes poſitivos in formula rationali $z^3 + 3m^2cz$, & habebis $8 + 3 \cdot 3 \cdot 2 = 26$, quod conditionem requiſitam adimplet, proindeque radix quæſita $z + m\sqrt{c} = 2 + \sqrt{3}$. Valor negativus -2 radicis cubicæ $-2 + \sqrt{3}$ extractioni ex binomio $-26 + 15\sqrt{3}$ inſervit; quod pro ceteris hujuſmodi caſibus ſemel præmonuiſſe ſufficiat.

II. Proponatur elicienda radix ex binomio cubico $12\frac{1}{2} + 11\sqrt{2}$. Reducatur ad idem nomen, eritque $\frac{25 + 22\sqrt{2}}{2}$. Ergo ſumpto numeratore, ſtabit $b = 22$, $c = 2$, & $\pm \sqrt[3]{\left(\frac{b - cm^3}{3m}\right)} = \pm \sqrt[3]{\left(\frac{22 - 2m^3}{3m}\right)}$. Ut hæc formula fiat rationalis, ſumi debet $m = 2$, & evadet $\pm \sqrt[3]{\left(\frac{22 - 16}{6}\right)} = \pm 1 = z$. Subſtitue ut in primo Exemplo, & prodibit $1 + 3 \cdot 4 \cdot 2 = 25$, quod cum hypotheſi optime convenit, quare $1 + 2\sqrt{2}$ erit binomii $25 + 22\sqrt{2}$ radix cubica; ſuperſt

igitur, ut hæc dividatur per numeri 2 radicem cubicam; adeoque jussa radix erit $\frac{1 + 2\sqrt{2}}{\sqrt[3]{2}}$:

III. Quæratur radix cubica ex binomio $3 + \frac{10}{3}\sqrt{-\frac{1}{3}}$. Ergo $b = \frac{10}{3}$, $c = -\frac{1}{3}$, & $\pm \sqrt[3]{\left(\frac{b - cm^3}{3m}\right)} = \pm \sqrt{\left(\frac{10 + m^3}{9m}\right)}$. Ut hæc fiat quantitas rationalis, ponatur $m = 2$; proveniet $\pm \sqrt{\left(\frac{10 + 8}{18}\right)} = \pm 1 = z$. Accepto signo negativo, substitue de more, & exurget $-1 + 4 = 3$, quod hypothesi favet, & radix imperata prodit $-1 + 2\sqrt{-\frac{1}{3}}$.

IV. Radix ex binomio cubico jam reducto $-80 - 72\sqrt{-3}$ extrahi jubeatur. Sumo binomium positivum $80 + 72\sqrt{-3}$. Ergo $b = 72$, $c = -3$. Hinc $\pm \sqrt{\left[\frac{b - cm^3}{3m}\right]} = \pm \sqrt{\left[\frac{72 + 3m^3}{3m}\right]} = \pm \sqrt{\left[\frac{24 + m^3}{m}\right]}$, ex quo colligo, esse $m = 2$, ac proinde $\pm \sqrt{\left[\frac{24 + m^3}{m}\right]} = \pm 4 = z$. Facta igitur cum signo negativo solita valorum substitutione, oritur $-64 + 144 = 80$, quod quum sit expositæ conditioni consentaneum, binomii positivi radix erit $-4 + 2\sqrt{-3}$; adeoque mutatis signis, propositi binomii cubici negativi radix imperata est $4 - 2\sqrt{-3}$.

V.

V. Detur investiganda radix ex binomio cubico $\sqrt{243}+\sqrt{242}$, quod redactum evadit $9\sqrt{3}+11\sqrt{2}$. Multiplicando, & dividendo per $3\sqrt{3}$, fit $\frac{81+33\sqrt{6}}{3\sqrt{3}}$. Ergo sumpto numeratore, erit $b=33$, $c=6$; hinc $\pm\sqrt{\left[\frac{b-cm^3}{3m}\right]}=\pm\sqrt{\left[\frac{11-2m^3}{m}\right]}$. Ut igitur hæc formula fiat rationalis, ponere opus est $m=1$, ex quo fit $\pm\sqrt{9}=\pm 3=z$. Itaque ut moris est subrogando, habetur $27+54=81$, quod cum hypothesi congruit, adeoque numeratoris $81+33\sqrt{6}$ radix erit $3+\sqrt{6}$; at hæc per $\sqrt{3}$ (quæ est denominatoris radix cubica) dividi debet; ergo $\frac{3+\sqrt{6}}{\sqrt{3}}=\sqrt{3}+\sqrt{2}$ est cubi $9\sqrt{3}+11\sqrt{2}$ radix quæsita.

VI. Expetatur radix cubica ex binomio $30\sqrt{-3}+37\sqrt{-1}$. Hoc ductum in cubum $-3\sqrt{-3}$ evadit $270+111\sqrt{3}$. Ergo $b=111$, $c=3$; quare $\pm\sqrt{\left[\frac{b-cm^3}{3m}\right]}=\pm\sqrt{\left[\frac{111-3m^3}{3m}\right]}=\pm\sqrt{\left[\frac{37-m^3}{m}\right]}$; necessario igitur erit $m=1$, adeoque $\pm 6=z$. Itaque substituendo ut supra, fit $216+54=270$, ut fert allata conditio. Hinc binomii $30\sqrt{-3}+37\sqrt{-1}$ radix erit $\frac{6+\sqrt{3}}{\sqrt{-3}}=-2\sqrt{-3}$

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$-\sqrt{-1}$, nam $\frac{6}{\sqrt{-3}} = \frac{6\sqrt{-3}}{-3} = -2\sqrt{-3}$, &

$\frac{\sqrt{3}}{\sqrt{-3}} = \frac{1}{\sqrt{-1}} = -\sqrt{-1}$.

VII. Elici jubeatur radix ex binomio cubico $2 + \sqrt{5}$. Ergo $b = 1$, $c = 5$, ex quo ſtatim perſpicio, radicem cubicam extrahi non poſſe, niſi per artificium aliquod id contingat. Igitur multiplico, & divido per 8, & numerator fit $16 + 8\sqrt{5}$, ex quo tento, an radix cubica extrahi poſſit. Ergo $b = 8$, $c = 5$, quare $\pm \sqrt[3]{\left(\frac{b - cm^3}{3m}\right)} = \pm \sqrt[3]{\left(\frac{8 - 5m^3}{3m}\right)}$; itaque neceſſe eſt ponere $m = 1$, ut formula evadat rationalis, quæ idcirco fit $= \pm 1 = z$. Subſtituo, & habeo $1 + 15 = 16$; quod quum conditionem includat, numeratoris radix cubica erit $1 + \sqrt{5}$; dividendo igitur per denominatoris 8 radicem cubicam, emergit binomii propoſiti radix $\frac{1 + \sqrt{5}}{2}$.

VIII. Invenienda ſit radix cubica binomii $5 + \frac{26}{9}\sqrt{3}$. Ergo $b = \frac{26}{9}$, $c = 3$; quare $\pm \sqrt[3]{\left[\frac{b - cm^3}{3m}\right]} = \pm \sqrt[3]{\left[\frac{26 - 27m^3}{27m}\right]}$, ex quo nihil inferre poſſum. Tento igitur, an datum binomium ſub alia forma poſſit repræſentari, & adverto, eſſe $\frac{26}{9}\sqrt{3}$ $=$

$= \frac{26}{3\sqrt{3}}$, proinde eidem binomio ſubſtituo $5 + \frac{26}{3\sqrt{3}}$, ſive $\frac{15\sqrt{3} + 26}{3\sqrt{3}}$; tum radici cubicæ ex numeratore extrahendæ me accingo. Ergo $b = 15$, $c = 3$; hinc $\pm \sqrt{\left[\frac{b - cm^3}{3m}\right]} = \pm \sqrt{\left[\frac{5 - m^3}{m}\right]}$. Ut igitur hæc formula fiat rationalis, eſſe debet $m = 1$, & ipſa tunc erit $= \pm 2 = z$. Facta igitur conſueta ſubſtitutione, provenit $8 + 18 = 26$, quod hypotheſi ſatisfacit. Ergo $2 + \sqrt{3}$ eſt dicti numeratoris radix cubica, adeoque binomii propoſiti habetur radix cubica $\frac{2 + \sqrt{3}}{\sqrt{3}} = 1 + \frac{2}{\sqrt{3}}$.

IX. Accedat binomium cubicum $5 + 3\sqrt{3}$, cujus radix debeat innoteſcere. Quoniam hic methodus ſimplex irrita evadit, multiplico binomium per 8, ſed fruſtra. Experior igitur, an numerus aliquis n in hoc binomium ductus eum reddere poſſit cubum, haud immemor, $\frac{3n - 3m^3}{3m}$, ſive $\frac{n - m^3}{m}$ eſſe debere quadratum. Quia vero unitas nihil immutat, ſumo $n = 2$, qui reddit quadratam formulam $\frac{n - m^3}{m}$, ſtatuendo $m = 1$; ergo pro binomio propoſito accipio

cipio $\frac{10+6\sqrt{3}}{2}$, & habeo $\pm\sqrt{\left(\frac{b-cm^3}{3m}\right)}=$ $\pm\sqrt{\left(\frac{6-3}{3}\right)}=\pm 1=z$; quare ſubſtituendo ut ſupra, reperio $1+9=10$, quod bene cedit; adeoque numeratoris $10+6\sqrt{3}$ radix cubica exurgit $=1+\sqrt{3}$. Hanc divido per factoris 2 radicem cubicam, & pro radice quæſita tandem obtineo $\frac{1+\sqrt{3}}{\sqrt[3]{2}}$.

X. Si denique ſcire aveas binomii $81+\sqrt{-2700}$ $=81+30\sqrt{-3}$ radicem cubicam, poſitis $b=30$, $c=-3$, $m=2$, reperies $x=\pm 3$; hinc per ſignum negativum habebis radicem quæſitam $-3+2\sqrt{-3}$. Sed ſi pro $81+30\sqrt{-3}$ ſumas binomium illi æquale $\frac{648+240\sqrt{-3}}{8}$, & ex numeratore radicem cubicam extrahas, ſtatuendo $b=240$, $c=-3$, $m=1$, obtinebis $x=\pm 9$; quare per valorem affirmativum $+9$ alia binomii cubici dati potieris radice $\frac{9+\sqrt{-3}}{2}$ a priori $-3+2\sqrt{-3}$ diverſa.

549. Methodus iſta, non tam ſevere, ut faciunt (quamvis tentando procedat) criminanda, reddi poſſet generalis pro eruendis ex dicto binomio $a+b\sqrt{c}$ radicibus quintis, ſeptimis, nonis, &c. ſed in calculos valde implicatos incideremus; quum enim pro

æqua-

æquatione ad portionem $b\sqrt{c}$ accipi debeant ſummæ terminorum, qui in locis paribus binomii $z+m\sqrt{y}$ ad poteſtatem quamlibet elati exiſtunt, liquet, reſolvendas eſſe æquationes ex derivativarum genere, quarum dimenſiones forent $2n$ (ſumpto n pro numero quovis integro affirmativo), nimirum $z^{2n}\sqrt{y} + pz^{2n-2}\sqrt{y} + qz^{2n-4}\sqrt{y} + \&c. = b\sqrt{c}$. Quamobrem præſtat via faciliori incedere ad eundem finem adſequendum. Sit igitur

550. Lemma. *Si binomium, cujus utraque pars ad quadratum evecta ſit rationalis, elevetur ad poteſtates, quarum index ſit numerus impar, fieri poteſt, ut ſumma terminorum tantummodo parium, vel tantummodo imparium, vel utrorumque ſit irrationalis.*

Dem. Sit binomium $x+y$, quod attollatur ad poteſtatem c, intelligendo per c numerum imparem; erit $(x+y)^c = x^c + \frac{c}{1}x^{c-1}y + \frac{c.\overline{c-1}}{1.2}x^{c-2}y^2 + \frac{c.\overline{c-1}.\overline{c-2}}{1.2.3}x^{c-3}y^3 + \&c.$ Termini igitur ſitus impares occupantes ſunt x^c, $\frac{c.\overline{c-1}}{1.2}x^{c-2}y^2$, &c. quorum ſummam voco A, ſitus autem pares tenentes ſunt $\frac{c}{1}x^{c-1}y$, $\frac{c.\overline{c-1}.\overline{c-2}}{1.2.3}x^{c-3}y^3$, &c. quorum

rum summam voco B. Sed quia c est numerus impar, si x sit rationalis, y irrationalis, A erit rationalis, B irrationalis; si contra x sit irrationalis, y rationalis, A erit irrationalis, B rationalis; si utraque x, & y sit irrationalis, erit etiam utraque A, B irrationalis, ut vel sola inspectione liquet; *Quod erat &c.*

551. *Coroll.* 1. Ergo si binomium $A+B$ referat potestatem quamlibet imparem, cujus radix $x+y$, sitque tam $A>B$, quam $x>y$, eandem quantitatem irrationalem minimam, quam A comprehendit, comprehendet quoque x, quamque B continet, etiam y continebit, ita ut si quantitas irrationalis, quam A includit, dignoscatur, innotescet etiam illa, quam x complectitur, & vicissim, quod pariter in B, & y accidere debere, apertum est. Sic si fuerit binomium ad quintam potestatem assurgens $\sqrt{184781088} \pm \sqrt{184773312}$; facta ad minimos terminos reductione, habetur $9612\sqrt{2} \pm 7848\sqrt{3}$, cujus radix $3\sqrt{2} \pm 2\sqrt{3}$; ubi statim conspicitur, quantitatem irrationalem $\sqrt{2}$ tam in binomii parte majori $\sqrt{184781088} = 9612\sqrt{2}$, quam in radicis parte majori $3\sqrt{2}$ contineri; itidemque quantitatem irrationalem $\sqrt{3}$ tam binomii parte minori $\sqrt{184773312} = 7848\sqrt{3}$, quam minori radicis parte $2\sqrt{3}$ comprehendi.

552. *Coroll.* 2. Si $(x+y)^c = A+B$, etiam $(x-y)^c = A-B$. Nam in potestate $(x-y)^c$ summa terminorum loca paria occupantium per incognitæ

gnitæ y potentias pares multiplicatorum eadem persistit, & signum mutant tantummodo termini in locis paribus existentes per ipsius y potentias impares multiplicati (65. n. IV.). Hinc si inveniatur

$x+y=\sqrt[c]{(A+B)}$, habebitur etiam $x-y=\sqrt[c]{(A-B)}$.

553. *Coroll.* 3. Si datum binomium quantitatibus omnibus integris constans radicem habeat divisore aliquo affectam, divisor iste esse non potest irrationalis. Si enim quotus $\frac{m\sqrt{x}\pm n\sqrt{y}}{\sqrt{p}}$ (in quo $\sqrt{x}$, $\sqrt{y}$, $\sqrt{p}$ sunt radicales inter se diversi ad minimam expressionem redacti) attollatur ad imparem quamlibet potestatem, semper $\sqrt{p}$ in denominatore invenietur; numerator itidem partim radicali $\sqrt{x}$, partim altero $\sqrt{y}$ in suis terminis afficietur. Sed $\sqrt{p}$ nullum ex radicalibus $\sqrt{x}$, $\sqrt{y}$ dividere [*ex hyp.*] potest; ergo potestas hæc impar quantitatibus omnibus integris constare nequit. Idem valet ratiocinium, si quotus ad potestatem imparem evehendus sit $\frac{x\pm n\sqrt{y}}{\sqrt{p}}$.

554. Propositio V. ***Radicem imparem quamlibet ex binomio extrahere, cujus uterque terminus ad quadratum evectus sit rationalis, & integer.***

Resolutio. Binomium datum sit $A+B$, cujus radicem imparem c, finge, esse $x+y$, nempe $x+y=\sqrt[c]{(A+B)}$, patet, binomii $A-B$ radicem c esse

esse debere $x-y$ (458.); quare $(x+y)(x-y) = \sqrt[c]{(A+B)} \times \sqrt[c]{(A-B)}$, seu $x^c-y^c = \sqrt[c]{(A^2-B^2)}$. Verum ad problema resolvendum expedit, ut A^2-B^2 sit potestas perfecta dimensionis c, quod quum non semper contingat, pono, binomium datum multiplicatum esse per indeterminatam m postmodum pro re nata determinandam, & pro $A+B$ sumo $mA + mB$, cujus radix sit semper $x+y$, videlicet $x+y = \sqrt[c]{(A+B)} \times \sqrt[c]{m}$, ita ut sit $A>B$, adeoque $x>y$. Igitur pars mA terminos qui situs impares, pars altera mB, qui pares occupant, repræsentabunt; ac proinde si A sit quantitas rationalis, B irrationalis, vel contra, vel si utraque sit irrationalis, etiam x erit rationalis, y irrationalis, vel contra, vel utraque irrationalis (550.), & $\frac{x+y}{\sqrt[c]{m}} = \sqrt[c]{(A+B)}$; ex quo sequitur, quod si ex $m(A+B)$ radicem extraho, etiam binomii $A+B$ radicem habere possum.

Quoniam vero $x+y = \sqrt[c]{m(A+B)}$, $x-y = \sqrt[c]{m(A-B)}$, fiet $x^2-y^2 = \sqrt[c]{m^2(A^2-B^2)}$. Sic si A^2-B^2 fuerit potestas perfecta, de qua quæritur, m^2 evadet $=1$, sin minus, quadratum aliquod repræsentabit ita determinandum, ut per ejus multiplicationem productum $m^2(A^2-B^2)$ ascendat ad pote-

potestatem requisitam, ex qua radix c extrahi possit, ubi animadverte, eo melius rem ex voto cessuram, quo minus fuerit quadratum m^2; hoc autem

quadratum, minori deficiente, erit $(A^2-B^2)^{c-1}$, nam ductum in A^2-B^2 potestatem gignet gradus c,

nimirum $(A^2-B^2)^c$.

Hisce stantibus, esto $\sqrt[c]{m^2(A^2-B^2)} = x^2-y^2 = n$, atque r exprimat quam proxime radicem c dati binomii, vel sit $\sqrt[c]{m(A+B)} = x+y = r$; sed $\frac{x^2-y^2}{x+y} = x-y$; ergo $x-y = \frac{n}{r}$ proxime. Addantur quantitates $x+y=r$, & $x-y=\frac{n}{r}$, ac fiet $2x = r+\frac{n}{r}$, adeoque $x = \frac{r+\frac{n}{r}}{2}$ proxime, quod agegatum statuo $=t$.

Hactenus ad valorem x proxime accessimus, sed memoriam revocandum, primum, & majorem omii terminum mA, ac proinde primum, & jorem radicis terminum x esse posse irrationales cit.], quare si x ponatur ad minimos terminos ictus, & minima quantitas radicalis, quam init, expressa per $\sqrt{p}$ dicatur s; erit $x = t\sqrt{p} = ts$ ar alt.

 =

$= \frac{r + \frac{n}{r}}{2}$, & $s = \frac{r + \frac{n}{r}}{2s}$.

Invento valore x, valor y facile haberi poteſt, etenim ob $y^2 = x^2 - (x^2 - y^2) = t^2 s^2 - n$, erit $y = \sqrt{t^2 s^2 - n}$; hinc ſi $m(A + B)$ ſit potentia perfecta, a qua radix imperata extrahi poſſit, radix iſta erit $ts + \sqrt{t^2 s^2 - n} = \sqrt[c]{(A + B)} \times \sqrt[c]{m}$, adeoque $\sqrt[c]{(A + B)} = \frac{x + y}{\sqrt[c]{m}} = \frac{ts + \sqrt{t^2 s^2 - n}}{\sqrt[c]{m}}$. *Quod erat* &c.

555. *Scholion.* Quum valor $r = x + y$ nonniſi per approximationem determinatus fuerit, ac proinde æquationes $x + y = r$, $x - y = \frac{n}{r}$ nonniſi per approximationem ſint veræ, ambigi ab aliquo poteſt, hanc methodum parum exactam eſſe, adeoque reiiciendam. Ut autem omnis dubitatio tollatur, oſtendendum eſt, ab erroribus, qui in duobus quantitatum $x + y$, $x - y$ valoribus ſic accipiendis committuntur, valorem quantitatis x incertum fieri non poſſe. Etenim quoniam $x = \frac{x + y + \frac{x^2 - y^2}{x + y}}{2} = \frac{r + \frac{n}{r}}{2}$, ponamus, errorem in valore quantitatis r determinando exprimi a fractione $\frac{a}{b}$, quæ minor ſemper erit

quam

n $\frac{1}{2}$; si enim r, utpote irrationalis, consistat

r duos numeros, vel limites $p+1$, & p, erit

$$\frac{1+\frac{n}{p+1}}{2}-\frac{(p+\frac{n}{p})}{2}=\frac{a}{b}=\frac{1}{2}-\frac{n}{p+pp}.$$ Er-

$$\frac{r+\frac{n}{r}}{2}=\frac{x+y\pm\frac{a}{b}+\frac{x^2-y^2}{x+y\pm\frac{a}{b}}}{2}.$$ Aufer x; super-

t $$\frac{y\pm\frac{a}{b}-\frac{y^2}{x+y\pm\frac{a}{b}}}{2}$$ seu $\pm\dfrac{\frac{2ay}{b}+\frac{a^2}{b^2}}{2x+2y\pm\frac{2a}{b}}$ exprimens

fferentiam inter verum valorem quantitatis x, illum, qui per approximationem adsumptus fuit. iquet autem, differentiam istam nunquam evade-

e posse $=\frac{1}{2}$; Si enim accipiatur signum $+$, erit

$$\frac{\frac{2ay}{b}+\frac{a^2}{b^2}}{2x+2y+\frac{2a}{b}}=\frac{\frac{2a}{b}\left(y+\frac{1}{2}\frac{a}{b}\right)}{2x+2y+\frac{2a}{b}};$$ Sed $\frac{a}{b}<\frac{1}{2}$,

adeoque $\frac{2a}{b}<1$; ergo numerator erit minor quam

$y+\frac{1}{2}\frac{a}{b}$, seu quam dimidium quantitatis $2y+\frac{a}{b}$,

& potiori jure minor quam dimidium denominatoris $2x+2y+\frac{2a}{b}$. Si sumatur signum $-$, differentia erit $-\frac{\frac{2ay}{b}+\frac{a^2}{b^2}}{2x+2y-\frac{2a}{b}} = -\frac{\frac{2a}{b}\left(y-\frac{1}{2}\frac{a}{b}\right)}{2x+2y-\frac{2a}{b}}$; & hic quoque ob $\frac{2a}{b}<1$, numerator minor erit quam dimidium quantitatis $2y-\frac{a}{b}$, ac multo magis minor quam dimidium denominatoris $2x+2y-\frac{2a}{b}$, sive quam $2x-\frac{a}{b}+2y-\frac{a}{b}$, quæ quantitas (quoniam $x>y$, & numeros integros uterque continere debet) semper erit dupla unitate major. Ergo differentia $x-\left(\frac{r+\frac{n}{r}}{2}\right)$, & multo magis $x-\left(\frac{r+\frac{n}{r}}{2s}\right)$ minor erit quam $\frac{1}{2}$. Sed de numeris integris agitur; ergo dicta approximatio valorem exactum ipsius x semper, si radicis extractio fieri poterit, exhibebit. *Quod erat* &c.

556.

556. *Coroll.* 1. Si *A* major binomii dati pars sit rationalis, quoniam x major radicis quæsitæ pars est pariter rationalis, erit $\sqrt{p}=s=1$, & radicis eruendæ formula evadet $\frac{t+\sqrt{t^2-n}}{\sqrt[c]{m}}$.

557. *Coroll.* 2. Similiter si ab æquatione $x+y$ auferas aliam $x-y=\frac{n}{r}$, habebis pro residuo $y=\frac{r-\frac{n}{r}}{2}=t$. Sit nunc s quantitas irrationalis minori dati binomii parti mB ad minimam expressionem redactæ inclusa; erit $y=ts=\frac{r-\frac{n}{r}}{2}$, unde (ob $x^2-y^2=n$, sive $x^2=y^2+n$) prodit $x^2=t^2s^2+n$, & $x=\sqrt{t^2s^2+n}$; signum enim negativum est inutile; quapropter $x+y=\sqrt{t^2s^2+n}+ts$, sive $\sqrt[c]{(A+B)}$ $=\frac{\sqrt{t^2s^2+n}+ts}{\sqrt[c]{m}}$; quæ quoque formula utilis esse potest. Demonstratio est eadem cum præcedenti (554.) mutatis tantum x in y, & vicissim.

558. *Coroll.* 3. Ab hujus problematis resolutione pendet, ut patet, exacta resolutio æquationum, quæ sub forma generali $x^{2c}+px^c=q$ (habent

 enim

enim $x = \sqrt[c]{-\frac{1}{2}p \pm \sqrt{q + \frac{1}{4}p^2}}$ comprehenduntur (309.), adeoque ex eadem trinomii quoque

$x^{2c} + px^c - q$ factores eliciuntur (504.); ex quo problematis præstantia percipitur.

559. *Monita.* Cautiones eædem, quas in monitis Prop. antec. (548.) enumeravimus, hic quoque cum quibusdam aliis locum obtinent; nimirum

(1.°) Si B fuerit in binomio quantitas negativa, etiam y in radice talis erit (552.); quare tam binomii $A + B$, quam radicis $x + y$ formula utrique casui inservit.

(2.°) Si binomii termini sint ambo negativi, sumantur affirmativi, & exacta operatione, signa in reperta radice c immutentur. Quod si primus binomii terminus, major scilicet, sit negativus, ac minor positivus, sumpto utroque affirmativo, major inventæ radicis terminus fiat negativus.

(3.°) Si binomii terminus alteruter sit fractus, fiat ad eundem denominatorem reductio, & numeratoris radix c inventa per denominatoris radicem c dividatur; sic numerator numeris integris, ut fert problematis conditio, constabit.

(4.°) Si binomium datum ad minimam expressionem reductum habeat sub signis radicalibus numeros fractos; ita prius reducatur, ut sub dictis signis numeros integros contineat, deinde tam ex numeratore quam denominatore radix imperata de

more

more extrahatur. Sic $3\sqrt{3}+7\sqrt{\frac{1}{2}}=\frac{3\sqrt{6}+7}{\sqrt{2}}$, multiplicando, ac dividendo per $\sqrt{2}$.

(5.°) Si datum binomium per quantitatem aliquam sit divisibile, divisionem institue antequam radicis c extractionem aggrediaris, ut operatio facilior reddatur. Sic si proponatur $\sqrt[3]{3993}+\sqrt[6]{17578125}$, divide per $\sqrt[3]{3}$; & quoniam $\frac{\sqrt[3]{3993}}{\sqrt[3]{3}}=\sqrt[3]{1331}=11$; $\frac{\sqrt[6]{17578125}}{\sqrt[3]{3}}=\frac{\sqrt[6]{17578125}}{\sqrt[6]{9}}=\sqrt[6]{1953125}=\sqrt{125}$; binomium propositum evadet $\sqrt[3]{3}(11+\sqrt{125})$ ex cujus utroque factore radix c seorsum elicienda.

(6.°) Si uterque binomii terminus sit irrationalis, expedit interdum majorem reddere rationalem; pro quo finge $A=\sqrt{m}$, $B=\sqrt{n}$; multiplicetur $\sqrt{m}+\sqrt{n}$ per $\sqrt{m^c}$, factum erit $\sqrt{m^{c+1}}+\sqrt{nm^c}$; sed $c+1$ est numerus par, quod c ponitur impar; ergo si statuatur $2p=c+1$, habetur $\sqrt{m^{c+1}}$

$+ \sqrt[c]{nm} = (\sqrt{m} + \sqrt{n})\sqrt{m^c} = m^p + \sqrt[c]{nn}$; quamobrem si hujus binomii quæratur radix c, hæc

erit $x + y = \sqrt[c]{(\sqrt{m} + \sqrt{n})\sqrt{m}^c} = \sqrt{m} \times \sqrt[c]{(A+B)}$, adeoque $\frac{x+y}{\sqrt{m}} = \sqrt[c]{(A+B)}$. id autem in radicibus ex poteſtatibus altioribus eruendis moleſtiorem ſæpe calculum adfert.

(7.°) Quod ſi primus terminus ſit irrationalis, alter rationalis, ac primus altero minor, primus ad minimam expreſſionem redactus fiat, ſi lubeat, rationalis, altero irrationali evadente.

(8.°) Quum ex binomio, cujus uterque terminus ad quadratum evectus ſit rationalis, radix impar quælibet (dummodo binomium ſit poteſtas perfecta, de qua agitur) inveſtigari poſſit; liquet, ex recenſito binomio radicem quancumque cum parem per Prop. II. (543.), tum imparem per præſentem educi poſſe, adeo ut ſi radicis quæſitæ index c ſit numerus par, & in radicis extractione per numeri illius paris dimidium ſucceſſive acceptum ad numerum imparem tandem deveniatur, tunc per præſentem Propoſitionem extractio abſolvetur. Sic radix quarta invenitur, ut vidimus (*l. c. Exempl. V.*) radicem quadratam bis extrahendo; radix ſexta habetur, prius radicem quadratam, deinde cubicam educendo; radix octava, quadratam ter eliciendo; radix nona, bis cubicam eruendo, & ſic deinceps.

Ce-

Ceterum debet prius animadverti, quod si fuerit e numerus par, & primus ac major binomii terminus irrationalis, frustra radicem quadratam invenire tentabis; quæcumque enim sit quantitas $\sqrt{x} \pm \sqrt{y}$ ad potestates pares elata, semper primus potestatis terminus esse debet rationalis, ut est manifestum.

Exempla.

I. Sit extrahenda radix cubica ex binomio $68 + \sqrt{4374}$. Erit igitur $A = 68$; $B = 4374$; quare $A^2 - B^2 = 4624 - 4374 = 250$. Ut habeas cubum, duc 4 in 250, ac proveniet 1000, cujus radix cubica $= 10$; hinc $m^2 = 4$, $m = 2$. Datum proinde binomium multiplico per 2, & obtineo $\frac{136 + \sqrt{17496}}{2}$, quod erit novum binomium proposito æquale, ex quo radix cubica elicienda. Ergo $mA = 136$; $mB = \sqrt{17496}$; $(A^2 - B^2) m^2 = 18496 - 17496 = 1000$, cujus radix cubica $10 = n$; $mB = \sqrt{17496} = 132$ proxime; $\sqrt[3]{m(A+B)} = \sqrt[3]{536} = 8$ proxime $= r$. Quamobrem $\frac{r + \frac{n}{r}}{2} = \frac{8 + \frac{10}{8}}{2} = \frac{74}{16} = 4$ proxime $= t$.

$= t$. Ergo $\frac{t + \sqrt{t^2 - n}}{\sqrt[3]{m}} = \frac{4 + \sqrt{16 - 10}}{\sqrt[3]{2}} =$

$\frac{4 + \sqrt{6}}{\sqrt[3]{2}}$ pro radice quæſita.

II. Sit binomium $\frac{27 + 11\sqrt{6}}{\sqrt{3}}$. Ergo $A = 27$;

$B = 11\sqrt{6} = \sqrt{726}$; $A^2 - B^2 = 729 - 726 = 3$. Ut igitur 3 fiat cubus, duc 9 in $A^2 - B^2 = 3$, ac fiet 27, cujus radix cubica $= 3 = m$. Itaque per 3 multiplicetur, ac dividatur datum binomium, & prodibit $\frac{81 + 33\sqrt{6}}{3\sqrt{3}}$ ad extrahendam radicem cubicam pro dato adſumendum. Ergo $mA = 81$; $mB = 33\sqrt{6} = \sqrt{6534}$; $m^2(A^2 - B^2) = 6561 - 6534 = 27$, cujus radix cubica $= 3 = n$; $m(A + B)$ $= 81 + 81$ proxime $= 162$; $\sqrt[3]{m(A+B)} = 5$ proxime $= r$; Hinc $\frac{r + \frac{n}{r}}{2} = \frac{5 + \frac{3}{5}}{2} = \frac{14}{5} =$ 3 proxime $= t$; quare $\frac{t + \sqrt{t^2 - n}}{\sqrt[3]{m}} = \frac{3 + \sqrt{9 - 3}}{\sqrt{3}}$

$= \frac{3 + \sqrt{6}}{\sqrt{3}} = \sqrt{3} + \sqrt{2}$ erit radix expetita.

III.

III. Sit binomium cubicum $\sqrt{243}+\sqrt{242} = 9\sqrt{3}+11\sqrt{2}$. Primum terminum secundo majorem libet reddere rationalem; itaque multiplico per $3\sqrt{3}$, & obtineo $\frac{81+33\sqrt{6}}{3\sqrt{3}}$. Hoc igitur binomium, quamvis sub diversa forma, idem est atque illud Exempli antecedentis, cujus radicem cubicam vidimus, esse $\sqrt{3}+\sqrt{2}$.

IV. Sit binomium cubicum $\sqrt{242}+12\frac{1}{2}$. Facta ad communem denominatorem reductione, provenit $\frac{\sqrt{968}+25}{2} = \frac{22\sqrt{2}+25}{2}$. Reddere libet primum terminum rationalem, multiplicando, & dividendo per $\sqrt{8} = 2\sqrt{2}$, & obtineo $\frac{88+\sqrt{5000}}{2 \cdot 2\sqrt{2}}$, quod idem est ac datum binomium, quodque accipio pro radice jussa eruenda. Ergo $A = 88$; $B = \sqrt{5000}$; $A^2-B^2 = 2744$, cujus radix cubica $= 14 = n$; quare $m^2 = 1$, $m = 1$, $\sqrt{5000} = 70$ proxime; hinc $m(A+B) = 158$, & $\sqrt[3]{m(A+B)} = 5$ proxime $= r$; ergo $\frac{r+\frac{n}{r}}{2} = \frac{5+\frac{14}{5}}{2} = \frac{5+3}{2}$ $= 4$

$=4$ proxime $=t$; adeoque $\frac{t+\sqrt{t^2-n}}{\sqrt[3]{m}}=4+\sqrt{2}$; & extrahendo radicem cubicam ex denominatore $2.2\sqrt{2}$, emergit $\frac{4+\sqrt{2}}{\sqrt[3]{2.\sqrt{2}}}=\frac{2\sqrt{2}+1}{\sqrt[3]{2}}$ pro desiderata binomii dati radice cubica.

V. Sit $\frac{13}{9}\sqrt{12}+5$. Utar formula Coroll. 2. sine prævia binomii reductione. Ergo $A=\frac{13}{9}\sqrt{12}$; $B=5$; $A^2=\frac{169}{81}.12=\frac{676}{27}$; $B^2=25=\frac{675}{27}$; $A^2-B^2=\frac{1}{27}$; $n=\frac{1}{3}$; $m=1$; $A+B=\frac{39}{9}+5=\frac{39+45}{9}=\frac{84}{9}=9$ proxime, cujus radix cubica 2 proxime $=r$. Est autem B rationalis; ergo $s=1$; præterea $\frac{r-\frac{n}{r}}{2s}=\frac{2-\frac{1}{6}}{2}=1-\frac{1}{12}=1$ proxime $=t$. Hinc $\sqrt{t^2s^2+n}+ts$ $=$

$= \sqrt{1+\frac{1}{3}}+1 = \sqrt{\frac{4}{3}}+1 = \frac{2}{\sqrt{3}}+1$; quæ est radix cubica optata.

Si quis calculum ineat, experietur quantum hæc formula pro radice ista cubica extrahenda, sit alia expeditior.

VI. Sit quintæ potestatis binomium $5\sqrt{5}+11 = \sqrt{125}+11$. Ergo $A = 125$; $B = 11$; $A^2-B^2 = 125-121 = 4$. Ut habeam numerum, qui ductus in 4 quintam referat potestatem, sumo $(A^2-B^2)^{c-1} = (A^2-B^2)^4 = 16 \times 16 = 256$, quem multiplico per 4, & obtineo 1024, cujus radix quinta $= 4$. Hinc $m^2 = 256$, $m = 16$. Itaque multiplico ac divido datum binomium per 16, & provenit

$\frac{80\sqrt{5}+176}{16}$ novæ formæ binomium dato æquale,

quod ad ejus radicem quintam extrahendam mihi propono. Ergo $mA = 80\sqrt{5}$; $mB = 176$, $m^2(A^2-B^2) = 1024$, cujus radix quinta $= 4 = n$; $80\sqrt{5}$

$= \sqrt{32000} = 181$ proxime; $\sqrt[5]{m(A+B)} = \sqrt[5]{181+176} = \sqrt[5]{357} = 3$ proxime $= r$; $s = \sqrt{5}$; quapropter $\frac{r+\frac{n}{r}}{2s} = \frac{3+\frac{4}{3}}{2\sqrt{5}} = \frac{13}{6\sqrt{5}} = 1$,

pro-

proxime $= r$; Ergo radix quæsita $\frac{rs + \sqrt{r^2 s^2 - n}}{\sqrt[c]{m}} = \frac{\sqrt{5}+1}{\sqrt[5]{16}}$.

VII. Sit ex binomio $29\sqrt{6} + 41\sqrt{3}$ quinta radix elicienda. Ergo $A = 29\sqrt{6} = \sqrt{5046}$; $B = 41\sqrt{3} = \sqrt{5043}$; $A^2 - B^2 = 5046 - 5043 = 3$. Multiplicetur 3 per 81, ut fiat quinta potestas 243, cujus radix quinta $= 3$; hinc $m^2 = 81$, $m = 9$. Itaque multiplico binomium datum per 9, & exurgit $9\sqrt{5046} + 9\sqrt{5043}$. Ergo $mA = 9\sqrt{5046}$; $mB = 9\sqrt{5043}$; $m^2(A^2 - B^2) = 408726 - 408483 = 243$, cujus radix quinta $= 3 = n$; $\sqrt{408726} = 641$ proxime, pariterque $\sqrt{408483} = 641$ proxime; quare $\sqrt[5]{m(A+B)} = \sqrt[5]{1282} = 4$ proxime $= r$; $mA = 29 \cdot 9\sqrt{6}$; ergo $s = \sqrt{6}$; ac proinde $\frac{r + \frac{n}{r}}{2s} = \frac{4 + \frac{3}{4}}{2\sqrt{6}} = \frac{19}{4\sqrt{24}} = \frac{19}{20} = 1$ proxime $= t$. Ergo $\frac{ts + \sqrt{t^2 s^2 - n}}{\sqrt[5]{m}} = \frac{\sqrt{6} + \sqrt{6-3}}{\sqrt[5]{9}} =$

$= \frac{\sqrt{6}+\sqrt{3}}{\sqrt[5]{9}}$, quæ eſt radix expetita.

VIII. Sit binomium $\frac{5042 + 2911\sqrt{3}}{5}$, ex quo radix ſeptima eruenda. Ergo $A = 5042$; $B = 2911\sqrt{3} = \sqrt{25421763}$; $A^2 - B^2 = 25421764 - 25421763 = 1 = m = n$; $B = \sqrt{25421763} = 5042$ proxime; quare $A+B = 10084$; $\sqrt[7]{(A+B)} = \sqrt[7]{10084} = 3$ proxime $= r$; Hinc $\frac{r + \frac{n}{r}}{2} = \frac{3 + \frac{1}{3}}{2} = 2$ proxime $= t$; ac tandem $\frac{t + \sqrt{t^2 - n}}{\sqrt[4]{m}} = 2 + \sqrt{3}$; adeoque $\frac{2+\sqrt{3}}{\sqrt[7]{5}}$ radix poſtulata.

CAPUT VII.

DE ÆQUATIONIBUS DERIVATIVIS, UBI DE ÆQUATIONIBUS PURIS, ET DE THEOREMATE COTESIANO.

560. D*Efinitio.* Æquationes omnes ſub hac forma comprehenſæ, nimirum $x^{mn} + px^{m(n-1)} + qx$

$qx^{m(n-2)} + rx^{m(n-3)} + sx^{m(n-4)} + \&c. = 0$, in quibus m, n sunt numeri integri affirmativi, sed m unitate major, *derivatæ*, seu *derivativæ* appellantur.

561. Propositio I. *Æquationes derivativæ in alias graduum minorum deprimi possunt.*

Dem. Esto $x^{mn} + px^{m(n-1)} + qx^{m(n-2)} + rx^{m(n-3)} + sx^{m(n-4)} + \&c. = 0$. Statue $z = x^m$; fiet $x^{mn} = z^n$; $x^{m(n-1)} = z^{n-1}$; $x^{m(n-2)} = z^{n-2}$, &c.; quare substituendo, nova prodit æquatio $z^n + pz^{n-1} + qz^{n-2} + rz^{n-3} + \&c. = 0$. Ergo &c.

562. *Scholion.* Ad æquationes derivativas pertinere aliquo modo etiam potest alia æquatio generalis $x^{m^n} + px^{m^{n-1}} + qx^{m^{n-2}} + rx^{m^{n-3}} + sx^{m^{n-4}} \ldots + P = 0$, in qua scilicet incognitæ x exponentes progressionem geometricam decrescentem servant, dummodo ultimus incognitæ index non sit unitas; hæc enim æquatio instar derivativarum deprimi potest; sit enim $m = 9$, $n = 2$; fiet $x^{81} + px^{27} + qx^9 + P = 0$. Pone $x^9 = z$; erit $x^{27} = z^3$; $x^{81} = z^{27}$. Substitue, & habebis $z^{27} + pz^9 + qz^3 + P = 0$. Fac iterum $y = z^3$; erit $y^3 = z^9$; $y^9 = z^{27}$; quare substituendo prodibit $y^9 + py^3 + qy + P = 0$, in qua $\sqrt[3]{y} = z$; sed $z = x^9$; ergo $\sqrt[3]{y} = x^9$; $y = x^{27}$; $\sqrt[27]{y} = x$;

$= x$; ubi apparet, sufficere ad maximam hujusmodi æquationum depressionem, ut incognitæ exponentes in serie geometrica progredientes per ultimum seriei terminum dividantur.

563. *Coroll.* 1. Hinc facilior interdum per hanc depressionem in æquationibus derivativis radicum extractio. Sit ex. gr. æquatio $x^{3m} - 3x^{2m} + 3x^m - 4 = 0$, quæ, posita $x^m = z$, migrat in sequentem $z^3 - 3z^2 + 3z - 4 = 0$. Ex hac si tertius terminus (statuendo $z = y + 1$) eliminetur; proveniet $y^3 ** - 3 = 0$, adeoque $y = \sqrt[3]{3}$, quare $z = 1 + y = 1 + \sqrt[3]{3}$; sed $z = x^m$; ergo $x^m = 1 + \sqrt[3]{3}$, ac demum $x = \sqrt[m]{1 + \sqrt[3]{3}}$. Similiter sit $x^{4n} - 6x^{2n} - 8x^n - 3 = 0$. Pone $z = x^n$; fiet $z^4 - 6z^2 - 8z - 3 = 0$; statue $z = y - 1$; habebis $y^4 - 4y^3 *** = 0$, unde habetur $y = 4$; sed $z = y - 1$; ergo $z = 4 - 1 = 3$; est autem $z = x^n$; ergo $x^n = 3$, ac proinde $x = \sqrt[n]{3}$.

564. *Coroll.* 2. Vicissim data æquatione, cujus radices innotescant, infinitæ dantur æquationes, quarum radices exprimi facillime poterunt. Esto $z^3 -$

$3z^2 + 3z - 4 = 0$, in qua vidimus, esse $z = 1 + \sqrt[3]{3}$. Si fiat $x^m = z$, substitutis valoribus, nova emerget æquatio $x^{3m} - 3x^{2m} + 3x^m - 4 = 0$, cujus radix $= x = \sqrt[m]{1 + \sqrt[3]{3}}$; & sic de reliquis.

565. Propositio II. *Cum in æquatione, in qua maximus incognitæ exponens est numerus par, termini per vices deficiunt, indicio est, tot paria tantummodo radicum æqualium sub signo diverso æquationem continere, quot dicti maximi exponentis dimidium indicat.*

Dem. Esto æquatio derivativa $x^{2n} + px^{2n-2} + qx^{2n-4} +$ &c. $= 0$. Pone $x^2 = y$; erit $x^{2n} = y^n$; $px^{2n-2} = py^{n-1}$; $qx^{2n-4} = qy^{n-2}$; &c. Quoniam termini [*ex hyp.*] per vices deficiunt; si fingamus, transformatæ radices esse a, b, c, &c. hæc transformata erit $y^n + py^{n-1} + qy^{n-2} +$ &c. $= 0$; ergo habemus $y = a$; sed $y = x^2$; ergo $x^2 = a$, & $x = \begin{matrix} +\sqrt{a} \\ -\sqrt{a} \end{matrix}$, & sic de ceteris radicibus; quare constat, propositam derivativam geminas habere radices æquales, sed signo diverso affectas. Sic æquationis $x^4 * - (a^2 + b^2) x^2 * + a^2b^2 = 0$ radices sunt $+a$, $-a$, $+b$, $-b$; & æquationis $x^6 * - (a^2 + b^2 - c) x^4 * -$ (a^2c

$(a^2c + b^2c - a^2b^2)x^2 * + a^2b^2c = 0$ radices sunt $+a$, $-a$, $+b$, $-b$, $+\sqrt{-c}$, $-\sqrt{-c}$. *Quod erat* &c.

566. *Coroll.* Data æquatione generali ordinata per incognitam x, quoniam x diversas radices indiscriminatim repræsentat, fiat $x' = x + z$; ergo z adæquat in transformata unam ex propositæ radicibus aliqua reliquarum detracta, nempe $z = x' - x$. Sint propositæ æquationis radices a, b, &c., erit $z = a - b$, nec non $z = a - c$, &c. quare transformata quæ diversa radicum paria contineat, terminis in locis paribus carebit, adeoque erit ex genere derivativarum, quas inter æquationes in Propositione indicatas locum habere, apertum est.

567. Propositio III. *In æquationibus derivativis tot termini post primum per vices deficiunt, quot unitates una dempta includit numerus ille constans, qui earundem æquationum exponentes multiplicat.*

Dem. Si numeri naturaliter progredientes multiplicentur per m, patet, eorum differentiam esse $= m$, & numeri, qui inter duos producti, seu novæ progressionis terminos deficiunt, erunt $= m - 1$. Sic si progressio 6, 5, 4 multiplicetur per 4, prodit 24, 20, 16, & numeri qui desunt inter terminos 24, & 20, vel inter 20, & 16, sunt tres unitates, seu $4 - 1$. Pari ratione si æquationum exponentes, qui progressionem arithmeticam naturalem descendentem servant, multiplicentur per m, unitates intermediæ deficientes erunt $m - 1$. Sed exponentium defectus indicat defectum terminorum in æquatione; ergo in æquationibus derivativis, qua-

rum forma $x^{mn} + px^{m(n-1)} + qx^{m(n-2)} + rx^{m(n-3)} + \&c. = 0$ terminorum deficientium numerus erit semper $= m - 1$. Sic si ponatur $m = 2, n = 2$; æquatio ista generalis evadet $x^4 * + px^2 * + q = 0$, in qua unus terminus per vices deficit ob $m - 1 = 1$; quod si statuatur $m = 3$, $n = 4$, dicta æquatio fiet $x^{12} ** + px^9 ** + qx^6 ** + rx^3 ** + s = 0$, in qua duo termini ob $m - 1 = 2$ per vices deficiunt, & sic semper; *Quod erat* &c.

568. Propositio IV. *Æquatio qualibet derivativa expressa per solitam formam* $x^{mn} + px^{m(n-1)} + qx^{m(n-2)} + \&c. = 0$ *habere non potest radices reales plus quam* 2n, *si* m *sit numerus par; non vero plus quam* n, *si* m *sit impar; ac proinde tunc erunt reliquæ imaginariæ.*

Dem. Quoniam sub hac generali æquatione etiam æquationes puræ, seu duobus tantum terminis constantes, seu cum $n = 1$, comprehenduntur, ab istis ordiar. Sit itaque

I. $x^m = \frac{a}{b}$, & m sit numerus par. Extracta radice m, fit $m = \pm \sqrt[m]{\frac{a}{b}}$, attollendo enim utrumque æquationis membrum ad potestatem m, fit rursus $x^m = \frac{a}{b}$; sed nulla alia quantitas realis præter $+ \sqrt[m]{\frac{a}{b}}$, vel

vel $-\sqrt[m]{\frac{a}{b}}$ ad poteſtatem m evecta evadere poteſt $\frac{a}{b}$; ergo duas tantum, nimirum $2n$, radices reales æquatio quævis pura vel ſimplex $x^m = \frac{a}{b}$ habere poteſt, ſi m ſit numerus par. Jam ponatur m numerus impar. Si ex $x^m = \frac{a}{b}$, vel $x^m = -\frac{a}{b}$, extrahi debeat radix m, ſemper fiet $x = \sqrt[m]{\frac{a}{b}}$ in primo caſu, & $x = \sqrt[m]{-\frac{a}{b}}$ in ſecundo, ſi enim utraque radix ad poteſtatem m rurſus evehatur, ſemper redibit in primo caſu æquatio $x^m = \frac{a}{b}$, in altero $x^m = -\frac{a}{b}$, nec alia quantitas realis id efficere poteſt. Sit perſpicuitatis gratia æquatio pura nullam quantitatem irrationalem involvens $x^5 = -32$; dico, nullum alium numerum præter -2 ad quintam poteſtatem elatum efficere poſſe -32. Si renuis, id præſtet numerus $-a\sqrt{b}$; at hic ad poteſtatem

quintam evectus evadit $-a^5b^2\sqrt{b}$, ubi numerus irrationalis necessario debet emergere, contra id quod positum fuerat; ergo nullus alius numerus realis præter -2 ad quintam potestatem elatus efficere potest -32. Hinc æquatio quævis pura, cujus exponens est numerus impar, unam tantum (nempe n) habere potest radicem realem.

II. Ponatur modo $n = 2$. Æquatio generalis evadet $x^{2m} + px^m + q = 0$. Ex hac more quadraticarum affectarum extrahatur radix. Si m sit numerus par, prodibit $x = \pm \sqrt[m]{-\frac{1}{2}p \pm \sqrt{q + \frac{1}{4}pp}}$ (309.), quæ formula quatuor (hoc est $2n$) nec amplius continere potest radices reales; at si m statuatur numerus impar, remanet semper $x = \sqrt[m]{-\frac{1}{2}p \pm \sqrt{q + \frac{1}{4}pp}}$, quare duas (hoc est n) nec amplius, comprehendere potest radices reales.

Fiat $n = 3$. Æquatio generalis transibit in $x^{3m} + px^{2m} + qx^m + r = 0$. Fiat $z = x^m$, proveniet $z^3 + pz^2 + qz + r = 0$. Hæc autem æquatio plus quam tres radices reales habere nequit; at ob $z = x^m$, ut incognitæ x habeantur valores, oportet ex dictis radicibus extrahere radicem m; ergo si m est numerus par, singulæ radices signo ambiguo affici debent,

bent, adeoque æquationis $x^{3m} + px^{2m} + qx^{m} + r = 0$ radices reales plus quam ſex eſſe nequeunt; ſin autem m ſit numerus impar, remanent tres ut ante, quare in primo caſu erunt $2n$, & n in ſecundo.

III. Si ſtatuatur $n = 5$; habemus $x^{5m} + px^{4m} + qx^{3m} + rx^{2m} + sx^{m} + t = 0$, quæ transformata evadit $z^5 + pz^4 + qz^3 + rz^2 + sz + t = 0$, cujus non plus quam quinque eſſe poſſunt radices reales. Sed ſi rurſus eliciatur radix m, ut incognitæ x valor obtineatur, radices reales non plus eſſe poterunt quam decem, ſi numerus m eſt par; remanent vero quinque, ſi fuerit impar.

IV. Generatim æquatio generalis depreſſa, ponendo $z = x^m$, evadit $z^n + pz^{n-1} + qz^{n-2} + rz^{n-3} + sz^{n-4} \ldots + t = 0$, in qua radicum omnium poſſibilium numerus eſt n. Ex ſingulis hiſce radicibus extrahatur radix m ad obtinendum valorem incognitæ x. Si m eſt numerus par, radices ſingulæ, quarum numerus n, propter ſignum ambiguum duplicantur (per n. I.), adeoque radicum realium in æquatione generali numerus plus eſſe non poteſt quam $2n$; at ſi m eſt numerus impar, radicum realium numerus idem remanet; & idcirco major eſſe nequit quam n. Sed numerus radicum, queis æquatio generalis conſtat, eſt mn; ergo reliquæ eſſe neceſſario debent imaginariæ, quarum numerus eſt

$mn - 2n$, si m est numerus par, & $mn - n$, si numerus m est impar. *Quod erat* &c.

Quoniam de æquationibus puris verba fecimus, quum unus ex earum factoribus esse possit æquatio derivativa, sequens haud inelegans hic etiam locum obtinet.

569. Propositio V. *Dato binomio* $a^m + b^l$, *polynomium invenire, quo in datum binomium ducto, alterum emergat binomium, in quo quantitas* a *habeat exponentem* n, *&* *quantitas* b *non sit per* a *multiplicata.*

Resolutio. Polynomium, seu multiplicatorem indagandum, finge, esse

$$a^{\alpha} b^{\gamma} + a^{\delta} b^{\varepsilon} + a^{\mu} b^{\nu} + a^{\omega} b^{\varphi} + \&c.$$

Hoc in datum binomium $a^m + b^l$ ducto, factum in duas series (A), (B) divisum sic expono

$$a^{m+\alpha} b^{\gamma} + a^{m+\delta} b^{\varepsilon} + a^{m+\mu} b^{\nu} + a^{m+\omega} b^{\varphi} + \&c. \ (A)$$

$$+ a^{\alpha} b^{l+\gamma} + a^{\delta} b^{l+\varepsilon} + a^{\mu} b^{l+\nu} + a^{\omega} b^{l+\varphi} + \&c. (B)$$

Sed in hoc facto esse [*per hyp.*] debet primus terminus $= a^n$; ergo erit $m + \alpha = n$, adeoque $\alpha = n - m$, & $\gamma = 0$. Præterea quum omnes termini intermedii se (ut fert problematis conditio) destruere debeant, duo necessario contingent, nempe (1.°) si datum binomium sit $a^m + b^l$, multiplicatoris termini in locis paribus existentes esse debent negativi, ac

ac proinde omnes feriei (*B*) termini negativi; contra vero si binomium datum sit $a^m - b^l$, omnes multiplicatoris termini esse debent affirmativi, quia tunc omnes feriei (*B*) termini evadunt ut ante negativi; (2.°) fieri debet

$$\begin{array}{lcl} m + \delta = \alpha & & \delta = \alpha - m = n - 2m \\ \quad \varepsilon = l + \gamma & & \varepsilon = l \\ m + \mu = \delta & \text{nimirum} & \mu = \delta - m = n - 3m \\ \quad \nu = l + \varepsilon & & \nu = 2l \\ m + \omega = \mu & & \omega = \mu - m = n - 4m \\ l \; + \nu = \varphi & & \varphi = 3l \\ \quad \&c. & & \&c. \end{array}$$

Valoribus hisce substitutis, polynomium quæsitum evadet $a^{n-m} \mp a^{n-2m}b^l + a^{n-3m}b^{2l} \mp a^{n-4m}b^{3l} + \&c.$ productum autem

$$\begin{array}{l} a^n \mp a^{n-m}b^l + a^{n-2m}b^{2l} \mp a^{n-3m}b^{3l} + \&c. \\ \qquad \pm a^{n-m}b^l - a^{n-2m}b^{2l} \pm a^{n-3m}b^{3l} - a^{n-4m}b^{4l} \pm \&c. \end{array}$$

ubi omnes termini se destruunt, exceptis primo & ultimo. Sed hic ultimus a potestatibus ipsius a libet esse debet; ergo posito p pro terminorum numero in polynomio, ultimus producti terminus erit $a^{n-pm}b^{pl}$; quare $n - pm = 0$, $p = \frac{n}{m}$, & $pl = \frac{nl}{m}$. Productū igitur polynomii, seu multiplicatoris in datum binomium

mium erit $a^n \pm b^{\frac{nl}{m}}$; videlicet $-b^{\frac{nl}{m}}$, ſi binomium datum ſit $a^m + b^l$, & ultimus multiplicatoris terminus cadat in locum parem; & $+b^{\frac{nl}{m}}$, ſi ultimus multiplicatoris terminus cadat in locum imparem, vel ſi datum binomium ſit $a^m - b^l$; in primo enim caſu multiplicatoris ſigna per vices procedunt, in altero ſunt omnia affirmativa.

570. *Coroll.* 1. Dato igitur binomio irrationali, inveniri poteſt polynomium, per quod ſi binomium datum multiplicetur, binomium emergat rationale. Quum enim dentur exponentes m, l, ſi pro n accipiatur minimus numerus integer, qui reddat $\frac{nl}{m}$ numerum integrum, obtinentur exponentes polynomii quæſiti.

571. *Coroll.* 2. Viciſſim dato facto $a^n \pm b^{\frac{nl}{m}}$, & factorum uno $a^m \pm b^l$, inveniri poteſt alter factor

$$a^{n-m} \mp a^{n-2m} b^l + a^{n-3m} b^{2l} \mp a^{n-4m} b^{3l} + \&c.$$

572. *Coroll.* 3. Si $m = l$, fit $\frac{l}{m} = 1 = n$.

573. *Coroll.* 4. Fractiones irrationales, quarum denominator eſt binomium irrationale, reduci poſſunt ad ſimpliciorem expreſſionem, inveniendo polyno-

nomium, per quod tam numeratore quam denominatore multiplicatis, denominator evadat rationalis; at generalem hac de re methodum jam exposui (113.n.II).

Exemplum I. Binomium datum esto $\sqrt{a} - \sqrt[3]{b}$, seu $a^{\frac{1}{2}} - b^{\frac{1}{3}}$. Erit $m = \frac{1}{2}$, $l = \frac{1}{3}$, & $\frac{l}{m} = \frac{2}{3}$; ergo $n = 3$. Præterea $n - m = 3 - \frac{1}{2} = \frac{5}{2}$; $n - 2m = 3 - 1 = 2$; $n - 3m = 3 - \frac{3}{2} = \frac{3}{2}$; $n - 4m = 3 - 2 = 1$; $n - 5m = \frac{1}{2}$; $n - 6m = 0$; & hic sistendum.

Itaque quinque sunt exponentes ipsius a, quare quinque inveniendi sunt exponentes ipsius b, nimirum $l = \frac{1}{3}$; $2l = \frac{2}{3}$; $3l = 1$; $4l = \frac{4}{3}$; $5l = \frac{5}{3}$. Polynomium igitur quæsitum erit

$$a^{\frac{5}{2}} + a^2 b^{\frac{1}{3}} + a^{\frac{3}{2}} b^{\frac{2}{3}} + ab + a^{\frac{1}{2}} b^{\frac{4}{3}} + b^{\frac{5}{3}},$$

quo in binomium datum $a^{\frac{1}{2}} - b^{\frac{1}{3}}$ ducto, prodit $a^3 - b^2$.

Exemplum II. Sit $x^r - 1 = 0$. Quoniam binomium simplex, ex quo æquatio ista coalescit, est $x - 1 = 0$, si r est numerus impar; & $x \pm 1 = 0$, si r est

eſt numerus par; $x^r - 1$ haberi poteſt pro facto, & $x \pm 1$ pro uno ex factoribus. Ut itaque factor alter inveniatur, fiat $a^n - b^{\frac{nl}{m}} = x^r - 1$; & $a^m \pm b^l = x \pm 1$. Hinc $m = 1$, $l = 0$, $n = r$; adeoque alterum factum quæſitum

$$x^{r-1} \mp x^{r-2} + x^{r-3} \mp x^{r-4} + \&c.$$

ubi ultimus terminus $x^{r-r} = x^0 = 1$.

574. *Scholion.* (1.°) Ponatur $r = 3$. Factor ob $x^3 - 1 = 0$ terminos omnes habebit affirmativos, nempe $x^2 + x + 1 = 0$, ex quo exurgit $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$. Tres igitur ſunt æquationis puræ cubicæ radices, una realis, ac duæ imaginariæ, nempe 1; $-\frac{1}{2} + \frac{1}{2}\sqrt{-3}$; $-\frac{1}{2} - \frac{1}{2}\sqrt{-3}$.

(2.°) Fiat $r = 4$. Si ſumatur $x - 1$, factor erit $x^3 + x^2 + x + 1 = 0$; ſi accipiatur $x + 1$, factor prodibit $x^3 - x^2 + x - 1 = 0$. At primus factor diviſus per $x + 1$, vel ſecundus per $x - 1$ dant $x^2 + 1 = 0$, ac proinde $x = \pm\sqrt{-1}$; ergo quarta unitatis poteſtas quatuor habet radices, duas reales, ac duas imaginarias, nimirum 1; -1; $+\sqrt{-1}$; $-\sqrt{-1}$.

(3.°) Sit $r = 5$. Quum unus factor ſit $x - 1$, alter erit $x^4 + x^3 + x^2 + x + 1 = 0$.

(4.°)

(4.°) Statuatur $x = 6$. Ex factore $x - 1 = 0$ habetur factum $x^5 + x^4 + x^3 + x^2 + x + 1 = 0$, & ex factore $x + 1 = 0$ factum $x^5 - x^4 + x^3 - x^2 + x - 1 = 0$. Sed tam primum factum divisum per $x + 1 = 0$, quam secundum per $x - 1 = 0$ præbent $x^4 + x^2 + 1 = 0$, ex quo educitur $x = \pm \sqrt{-\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}}$; ergo sexta unitatis potestas sex habet radices, duas reales, & quatuor imaginarias, nempe $+1$; -1; $+\sqrt{-\frac{1}{2} + \frac{1}{2}\sqrt{-3}}$; $-\sqrt{-\frac{1}{2} + \frac{1}{2}\sqrt{-3}}$; $+\sqrt{-\frac{1}{2} - \frac{1}{2}\sqrt{-3}}$; $-\sqrt{-\frac{1}{2} - \frac{1}{2}\sqrt{-3}}$. Et sic de ceteris.

Occasionem hic nanciscor celeberrimi atque utilissimi Theorematis Cotesiani a Mathematicis præclarissimis diversimode jamdudum demonstrati, facilem rursus & ad Tironum captum accommodatam exhibendi demonstrationem; sed prius expedit præmittere.

575. Lemma. *Binomium* $1 \pm x^n$ *in divisores secundi gradus resolvere.*

Resolutio I. Statue $1 - x^n = 0$ (504.), & esto n numerus par. Quoniam exceptis 1, & -1, radices omnes sunt imaginariæ (liquet enim, nullam aliam quantitatem realem sumptam pro x efficere posse $x^n = 1$), reducantur istæ ad formulam $a + b\sqrt{-1}$ (132.), acceptis quantitatibus realibus a, b positivis aut

aut negativis prout opportunum fuerit, atque etiam $a = 0$. Æquationis igitur $x^n = 1$ radices imaginariæ erunt $a + b\sqrt{-1}$, $a - b\sqrt{-1}$, vel $-a - b\sqrt{-1}$, $-a + b\sqrt{-1}$; $c + d\sqrt{-1}$, $c - d\sqrt{-1}$, vel $-c - d\sqrt{-1}$, $-c + d\sqrt{-1}$, &c. (440.); sed productum primi paris adæquat productum paris secundi, &c.; ergo pro divisoribus simplicibus binomii $1 - x^n$ habemus quantitates $1 - x$; $1 + x$; $x - a - b\sqrt{-1}$; $x - a + b\sqrt{-1}$; $x + c + d\sqrt{-1}$; $x + c - d\sqrt{-1}$; &c. sive si paria proxima invicem ducantur, binomii $1 - x^n$ divisores quadratici erunt $1 - x^2$; $x^2 - 2ax + a^2 + b^2$; $x^2 + 2cx + c^2 + d^2$; &c. quorum numerus $= \frac{n}{2}$. *Quod erat primum.*

II. Posito n numero impari, quum binomii $1 - x^n$ radices omnes præter $+1$ sint imaginariæ, ejus divisores erunt $1 - x$, $x^2 - 2ex + e^2 + f^2$; $x^2 + 2gx + g^2 + h^2$; &c. quorum numerus $\frac{n-1}{2}$. *Quod erat secundum.*

III. Accedat nunc $x^n + 1 = 0$, sitque n numerus par; fiet $x^n = -1$, adeoque omnes quantitatis -1 radices nulla excepta erunt imaginariæ, ac proinde dicti binomii $1 + x^n$ divisores exprimentur per-

per $x^2 - 2px + p^2 + q^2$; $x^2 + 2rx + r^2 + s^2$; &c. quorum numerus $\frac{n}{2}$. *Quod erat tertium.*

IV. Quod si n numerus impar adsumatur, æquatio $x^n = -1$, radice reali -1 excepta, radices omnes habebit imaginarias; ex quo fit, ut binomii $1 + x^n$ in dicta hypothesi divisores sint $x + 1$; $x^2 - 2kx + k^2 + l^2$; $x^2 + 2tx + t^2 + v^2$; &c. quorum numerus $\frac{n-1}{2}$. *Quod erat quartum.*

576. *Coroll.* 1. Ergo in utroque casu n paris vel imparis radices imaginariæ in numero semper pari deprehenduntur, ut jam docuimus (423. 440.).

577. *Coroll.* 2. Habemus $x = a + b\sqrt{-1}$, $x = a - b\sqrt{-1}$, &c.; sed $a + b\sqrt{-1} = \frac{a^2 + b^2}{a - b\sqrt{-1}}$, & $a - b\sqrt{-1} = \frac{a^2 + b^2}{a + b\sqrt{-1}}$, & ita porro; ergo etiam

$$\frac{(a^2 + b^2)^n}{(a - b\sqrt{-1})^n}, \text{ nec non } \frac{(a^2 + b^2)^n}{(a + b\sqrt{-1})^n} = \pm 1.$$

Hinc multiplicando per $(a + b\sqrt{-1})^n = \pm 1$, fit $(a^2 + b^2)^n = 1$, pariterque $a^2 + b^2 = 1$, $c^2 + d^2 = 1$, $g^2 +$

$g^2+h^2=1$, &c. quod indicat, quantitates istas sumendas esse in circulo, ita ut a sit cosinus, b sinus, &c. ex quo fluit

578. *Constructio.* Radio $SA=1$ (*Fig.* 22.) descripto circulo $AMBE$, ac sumpto super eodem radio puncto quolibet P, fiat $SP=x$, $SD=a$, $SH=c$, adeoque excitatis ex punctis D, H ad circumferentiam normalibus DM, HN, erit $DM=b$, $HN=d$, nam $\overline{SD}^2+\overline{DM}^2=\overline{SA}^2=1=\overline{SH}^2+\overline{HN}^2$, & ita semper. Ductis præterea PM, PN, quoniam $DP=a-x$, erit $\overline{PM}^2=a^2-2ax+x^2+b^2=x^2-2ax+1$, quod etiam eveniet, si punctum D cadat inter puncta P, S. Similiter ob $PH=x+c$, erit $\overline{PN}^2=x^2+2cx+1$. Pari modo reperientur valores, quos habent quadrata omnium rectarum ex puncto P ad circumferentiam ductarum. Sed quia productis MD, NH ad circumferentiam in punctis E, F, actisque PE, PF, habemus $PM=PE$, $PN=PF$, erit $PM.PE=\overline{PM}^2=x^2-2ax+1$, itidemque $PN.PF=\overline{PN}^2=x^2+2cx+1$, & sic de reliquis. Quum itaque sint $AP=1-x$, $AH=AS+SP=1+x$, liquet, in productis $AP.AH$, $PM.PE$, $PN.PF$, &c. inclusos esse binomii $1-x^n$, seu $\overline{AS}^n-\overline{SP}^n$ divisores, facto n nu-

numero pari. Quomodo autem cosinus sint pro re nata accipiendi, ex jam demonstratis (216. 218.) colligitur.

579. Propositio VI. *Si circuli* (Fig. 23. 24.) *circumferentia, cujus centrum* S, *dividatur in partes æquales* 2n, *&* *ex puncto quolibet* P *sumpto in diametro agantur ad omnia divisionum puncta recta* PA, Pa, PB, Pb, PD, Pd, &c.

dico, fore $\overline{SA}^n - \overline{SP}^n = PA.PB.PD.PE.PF$ &c.

& $\overline{SA}^n + \overline{SP}^n = Pa.Pb.Pd.Pe.Pf$ &c.

Dem. I. Radio $SA = 1$ describe circulum (*Fig.* 22.), in quo si statuatur arcus $AM = 0$, vel $= C$, circumferentiæ divisio in partes æquales incipiet in A (218.), ut in *Fig.* 23. ubi partes æquales, quarum numerus $2n$, posito n pari, sint Aa, aB, Bb, Dd, dE, &c. Ob $AM = 0$, vel $= C$, series, quæ huc pertinet, erit $0, \frac{2C}{2n}, \frac{4C}{2n}, \frac{6C}{2n}, \frac{8C}{2n}$, &c. (219.), cujus ultimus arcus desinit in A, ut ex hypothesi constat. Sed $Aa = \frac{1}{2m}C$ (si enim ex. gr. sit C divisa in partes 2.6, arcus Aa erit totius pars duodecima, seu $Aa = \frac{C}{2.6}$), adeoque $AB = 2Aa = \frac{2C}{2n}$, & arcus $ABD = \frac{4C}{2n}$ &c. Ergo arcus

hoc in casu adsumendi erunt AB, ABD, ABE, ABF, &c. quorum cosinus esse nequeunt nec plus nec minus quam n (216.). Hisce præjactis, accipiatur in radio SA quodvis punctum P, ductisque rectis PB, PD, PE, PF, &c. fiat $SP = x$. Quoniam adsunt cosinus tam circumferentiæ (nimirum AS), quam semicircumferentiæ (nempe ES), pro reliquorum arcuum cosinubus statue arcus AB cosinum $= a$, arcus ABD cosinum $= c$, &c. Habebimus $AP = 1 - x$, $PE = AS + SP = 1 + x$, quibus invicem ductis, nec non rectis æqualibus PB, PG, & PD, PF, &c. inter se multiplicatis, exurgent facta $1 - x^2$, $x^2 - 2ax + 1$, $x^2 \mp 2cx + 1$, &c. (578.). Sed quantitates istæ componunt divisores omnes binomii $1 - x^n$ (575. n. I.); ergo $1 - x^n = \overline{AS}^n - \overline{SP}^n = PA.PB.PD.PE.PF$ &c.; *Quod erat primum.*

II. Retenta tam hypothesi n paris, quam circumferentiæ divisione in partes æquales $2n$, si pro primo arcu sumatur semicircumferentia $ABE = \frac{1}{2}C$, tunc revocanda est series $\frac{C}{2n}$, $\frac{3C}{2n}$, $\frac{5C}{2n}$, $\frac{7C}{2n}$, &c. (220.). Sed $Aa = \frac{C}{2n}$, $Ab = \frac{3C}{2n}$, &c. Ergo arcus seligendi sunt Aa, ABb, ABd, ADe, ADf, &c. propterea ductis Pa, Pb, Pd, Pe, Pf, &c. diametri portiones AP, PE hic locum in supputatione, ut in casu expo-

posito, non obtinent ob cosinuum tam circumferentiæ, quam semicircumferentiæ defectum, quod ex arcuum progressu statim apparet, ad quorum tantummodo terminos a, b, d, e, f, &c. rectæ ex P ducuntur. Itaque si primæ divisionis arcum Aa statuamus habere cosinum $= e$, & arcus Ab cosinum $= k$, &c. facta $Pa.Pg, Pb.Pf$, &c. erunt $x^2 - 2ex + 1$, $x^2 \mp 2kx + 1$, &c. Sed hujusmodi facta continent divisores omnes binomii $1 + x^n$ (575. n. III.); ergo $1 + x^n = \overline{AS}^n + \overline{SP}^n = Pa.Pb.Pd.Pe.Pf$ &c. *Quod erat secundum.*

III. Sit modo n numerus impar. Posito primo arcu $= 0$, vel $= C$, adeunda est series 0, $\frac{2C}{2n}$, $\frac{4C}{2n}$, $\frac{6C}{2n}$, &c. (219.). Quum autem (*Fig.* 24.) sit arcus $Aa = \frac{C}{2m}$, erit arcus $AB = \frac{2C}{2m}$, arcus $ABD = \frac{4C}{2m}$ &c. Arcus igitur accipiendi erunt, $AB, ABD, ABDE$, &c. ubi constat, arcum ABd, seu semicircumferentiam in censum venire non posse, posse autem circumferentiam. Positis igitur arcus AB cosinu $= g$, arcus ABD cosinu $= h$, &c.; facta $PB.PF, PD.PE$, &c. erunt $x^2 - 2gx + 1$, $x^2 \mp$

$x^2 \mp 2bx + 1$, &c. quibus si addatur $AP = 1 - x$, habemus $1 - x, x^2 - 2gx + 1, x^2 \mp 2bx + 1$, &c. Sed istæ quantitates efficiunt divisores omnes binomii $1 - x^n$ (575. n. II.); Ergo $1 - x^n = \overline{AS}^n - \overline{SP}^n = PA . PB . PD . PE$ &c. *Quod erat tertium.*

IV. Manentibus iisdem quæ in num. antec. si pro primo arcu statuatur semicircumferentia $ABd = \frac{1}{2} C$ (quæ idcirco evadet divisa in partes æquales sub numero impari n), confugiendum erit ad seriem $\frac{C}{2n}, \frac{3C}{2n}, \frac{5C}{2n}, \frac{7C}{2n}$, &c. (220.). Sed arcus $Aa = \frac{C}{2n}$, arcus $Ab = \frac{3C}{2n}$, &c.; ergo arcus sumendi erunt Aa, Ab, Ad, Ae, &c.; ex quo colligitur, ultimum arcum (puta $ABDEf$) nunquam posse ad punctum A pervenire, adeoque diametri portionem AP extra supputationem remanere, alteram vero portionem Pd esse admittendam. Itaque si arcus Aa cosinus ponatur $= p$, arcus Ab cosinus $= q$, &c. facta $Pa . Pf, Pb . Pe$, &c. erunt $x^2 - 2px + 1, x^2 \mp 2qx + 1$, &c. quibus addita $Pd = x + 1$, emerget series $x + 1, x^2 - 2px + 1, x^2 \mp 2qx + 1$, &c. Sed hujusmodi quantitates sunt divisores binomii $1 + x^n$ (575. n. IV.); ergo $1 + x^n = \overline{AS}^n$

$\overline{AS}^n + \overline{SP}^n = Pa.Pb.Pd.Pe$ &c. *Quod erat quartum.*

580. *Coroll.* 1. In hypothesi n paris productum ex omnibus rectis, quae ex puncto P aguntur ad omnia divisionum puncta cum circuli circumferentia dividitur in partes aequales $2n$, nimirum $PA.Pa.PB.Pb.PD.Pd.PE.Pe$ &c. (*Fig.* 23.), adaequant $(1-x^n)(1+x^n) = 1-x^{2n} = \overline{AS}^{2n} - \overline{SP}^{2n}$. Id colligitur ex hujus demonstrationis num. I. & II. In hypothesi pariter n imparis idem productum, reliquis stantibus, adaequat $(1-x^n)(1+x^n) = 1-x^{2n} = \overline{AS}^{2n} - \overline{SP}^{2n}$. Id infertur ex hujus demonstrationis num. III. & IV.

581. *Coroll.* 2. Si omnium rectarum, quae ex puncto P ductae sint (tam in casu num. I. & III. quam in casu num. II. & IV.) ad circumferentiam, sumantur quadrata; productum ex omnibus hisce quadratis erit aequale formulae $x^{2n} \pm 2x^n + 1 = \overline{PA}^{2n} \pm 2\overline{AS}^n . \overline{PA}^n + \overline{AS}^{2n}$.

582. *Coroll.* 3. Per divisionem igitur arcus circularis inveniuntur factores binomii $\pm x^n \pm r^n$, ponendo radium $r = 1$. Pariterque si statuatur $\overline{AS}^n$

O 3 =

$= \frac{1}{2} p$, & $\overline{AS}^{2n} = q$, fiatque $AS = 1$, pro $\overline{PA}^{2n}$

$\pm 2 \overline{AS}^{n} . \overline{PA}^{n} + \overline{AS}^{2n}$ emerget $\overline{PA}^{2n} \pm p . \overline{PA}^{n} +$. q, five $x^{2n} \pm px^{n} + q$, cujus factores trinomii fecundi gradus numero n per folam arcus circularis divifionem poffunt adfignari.

CAPUT VIII.

DE ÆQUATIONIBUS CUBICIS, EARUMQUE RESOLUTIONE.

583. ÆQuationes cubicas refolvendas ita prius tractatas effe ponimus, ut non folum fractionibus & radicalibus, fed fecundo etiam termino careant, quæ *reductæ* appellantur.

584. Quælibet æquatio cubica, cui deeft fecundus terminus, ad quatuor fpecies fequentes referri poteft, nempe

1. $x^3 + 3px + 2q = 0$
2. $x^3 + 3px - 2q = 0$
3. $x^3 - 3px - 2q = 0$
4. $x^3 - 3px + 2q = 0$

quæ tamen omnes ad unam poffunt per compendium reduci, ut exempla mox adferenda demonftrabunt.

585. Propofitio I. *Æquationes cubicas quafcumque refolvere.*

Re-

Resolutio. Sit æquatio generalis $x^3 - 3px - 2q = 0$. Fiat $x = y + z$; erit $x^3 = y^3 + 3y^2z + 3yz^2 + z^3$; sive $x^3 = y^3 + 3yz\,(y+z) + z^3$; adeoque $x^3 = y^3 + 3yzx + z^3$; ac proinde $x^3 - 3xyz - y^3 - z^3 = 0$, quæ æquatio est ejusdem formæ perinde ac generalis proposita. Instituta igitur terminorum homogeneorum comparatione (478.), habetur $-3yz = -3p$; $yz = p$; $-y^3 - z^3 = -2q$; $y^3 + z^3 = 2q$; & multiplicando per y^3, prodibit $y^6 + y^3z^3 = 2qy^3$. Sed ob $yz = p$, est $y^3z^3 = p^3$; ergo $y^6 + p^3 = 2qy^3$, seu $y^6 - 2qy^3 + p^3 = 0$; æquatio nimirum derivativa secundi gradus ex dictis (309.) facile resolubilis; quapropter invenietur $y^3 = q \pm \sqrt{(q^2 - p^3)}$, & radicem cubicam extrahendo, $y = \sqrt[3]{q \pm \sqrt{(q^2 - p^3)}}$; unde nullo negotio elicitur valor quantitatis z; quum enim sit $2q = y^3 + z^3$, erit $z^3 = 2q - y^3 = 2q - q \mp \sqrt{(q^2 - p^3)} = q \mp \sqrt{(q^2 - p^3)}$; hinc $z = \sqrt[3]{q \mp \sqrt{(q^2 - p^3)}}$; sed $x = y + z$; ergo $x = \sqrt[3]{q \pm \sqrt{(q^2 - p^3)}} + \sqrt[3]{q \mp \sqrt{(q^2 - p^3)}}$; ex quo infertur, valorem x nonnisi unicum esse posse, quia cum radicalis $\sqrt[3]{q + \sqrt{(q^2 - p^3)}}$ valorem y exprimens est positivus, ille, qui valorem z designat, est negativus, & vicissim, propterea eadem signorum remanet permutatio, ut concludi possit, esse $x = \sqrt[3]{q + \sqrt{(q^2 - p^3)}} + \sqrt[3]{q - \sqrt{(q^2 - p^3)}}$, qui erat canon generalis inveniendus pro quacum-

que æquatione cubica secundo termino carente resolvenda, dummodo æquationis specialis termini cum terminis homologis formulæ generalis rite comparentur, habita semper ratione signorum.

Aliter. Esto ut ante $x^3-3px-2q=0$ æquationes omnes cubicas cum suis signis repræsentans. Fiat $x=y+\frac{p}{y}$, quod cum hypothesi antecedente conspirat, quum enim ibi repertum fuerit $yz=p$, stabit $z=\frac{p}{y}$, adeoque $x=y+z=y+\frac{p}{y}$; erit

$$\left.\begin{array}{l} x^3=y^3+3py+\frac{3p^2}{y}+\frac{p^3}{y^3} \\ -3px=\quad -3py-\frac{3p^2}{y} \\ -2q=\qquad\qquad\qquad -2q \end{array}\right\}=0$$

$0=y^3 \quad \cdot \quad \cdot \quad -2q+\frac{p^3}{y^3}$. Multiplica per y^3; proveniet $y^6-2qy^3+p^3=0$. Extrahe radicem cubicam; fiet $y=\sqrt[3]{q\pm\sqrt{(q^2-p^3)}}$. Sed $x=y+\frac{p}{y}$; ergo $x=\sqrt[3]{q\pm\sqrt{(q^2-p^3)}}+\frac{p}{\sqrt[3]{q\pm\sqrt{(q^2-p^3)}}}$.

Sed

Sed $\frac{p}{\sqrt[3]{q+\sqrt{(q^2-p^3)}}} = \sqrt[3]{q-\sqrt{(q^2-p^3)}}$, & vicissim $\frac{p}{\sqrt[3]{q-\sqrt{(q^2-p^3)}}} = \sqrt[3]{q+\sqrt{(q^2-p^3)}}$ (113. n. II.); Ergo quoniam vel sumendo in signo ambiguo $\pm$ signum superius, vel inferius, semper (facta etiam per denominatores multiplicatione) prodit $x = \sqrt[3]{q+\sqrt{(q^2-p^3)}} + \sqrt[3]{q-\sqrt{(q^2-p^3)}}$, concludi potest, hanc formulam canonem exhibere generalem pro quacumque æquatione cubica secundo termino experte resolvenda. *Quod erat* &c.

586. *Scholion* I. Formula generalis, quam attulimus (quæque *Cardani Regula* nuncupatur, quod eam ille primus publicavit, quamvis præclarum ejusdem inventum Scipioni Ferreo Bononiensi, ut Cardanus ipse testatur, debeamus) unicam præbet æquationis cubicæ radicem. Ut autem reliquæ duæ habeantur, quoniam in primæ Resolutionis transformata $x^3-3yzx-y^3-z^3=0$ scimus, esse $x=y+z$, fiat per $x-y-z$ divisio, & pro quoto emerget $x^2+(y+z)x+y^2+z^2-yz=0$, sive $x^2+(y+z)x=yz-y^2-z^2$; quare $x=\pm\frac{1}{2}(y-z)\sqrt{-3}-\frac{1}{2}(y+z)$, ubi reliquæ duæ radices includuntur;

fa-

factis siquidem valorum $y = \sqrt[3]{q + \sqrt{(q^2 - p^3)}}$, $z = \sqrt[3]{q - \sqrt{(q^2 - p^3)}}$ substitutionibus prodit

$$(1.°)\ x = \frac{1}{2}(y - z)\sqrt{-3} - \frac{1}{2}(y + z) =$$

$$\frac{1}{2}\sqrt[3]{q + \sqrt{(q^2 - p^3)}}\sqrt{-3} - \frac{1}{2}\sqrt[3]{q - \sqrt{(q^2 - p^3)}}\sqrt{-3}$$

$$-\frac{1}{2}\sqrt[3]{q + \sqrt{(q^2 - p^3)}} - \frac{1}{2}\sqrt[3]{q - \sqrt{(q^2 - p^3)}} =$$

$$\sqrt[3]{q + \sqrt{(q^2 - p^3)}} \times \frac{-1 + \sqrt{-3}}{2} + \sqrt[3]{q - \sqrt{(q^2 - p^3)}} \times \frac{-1 - \sqrt{-3}}{2}.$$

$$(2.°)\ x = -\frac{1}{2}(y - z)\sqrt{-3} - \frac{1}{2}(y + z) =$$

$$-\frac{1}{2}\sqrt[3]{q + \sqrt{(q^2 - p^3)}}\sqrt{-3} + \frac{1}{2}\sqrt[3]{q - \sqrt{(q^2 - p^3)}}\sqrt{-3}$$

$$-\frac{1}{2}\sqrt[3]{q + \sqrt{(q^2 - p^3)}} - \frac{1}{2}\sqrt[3]{q - \sqrt{(q^2 - p^3)}} =$$

$$\sqrt[3]{q + \sqrt{(q^2 - p^3)}} \times \frac{-1 - \sqrt{-3}}{2} + \sqrt[3]{q - \sqrt{(q^2 - p^3)}} \times \frac{-1 + \sqrt{-3}}{2}.$$

Eas-

Easdem formulas sic quoque possumus obtinere. Quum quævis æquatio cubica tres habeat radices; quantitatum singularum y, z triplex erit valor, nimirum $y = \sqrt[3]{q+\sqrt{(q^2-p^3)}}$; $y = \sqrt[3]{q+\sqrt{(p^2-q^3)}} \times \frac{-1+\sqrt{-3}}{2}$; $y = \sqrt[3]{q+\sqrt{(q^2-p^3)}} \times \frac{-1-\sqrt{-3}}{2}$ (574. n. I.); ac similiter $z = \sqrt[3]{q-\sqrt{(q^2-p^3)}}$; $z = \sqrt[3]{q-\sqrt{(q^2-p^3)}} \times \frac{-1-\sqrt{-3}}{2}$; $z = \sqrt[3]{q-\sqrt{(q^2-p^3)}} \times \frac{-1+\sqrt{-3}}{2}$; quapropter incognitæ x triplex valor erit ut supra, videlicet

(1.°) $x = \sqrt[3]{q+\sqrt{(q^2-p^3)}} + \sqrt[3]{q-\sqrt{(q^2-p^3)}}$.

(2.°) $x = \sqrt[3]{q+\sqrt{(q^2-p^3)}} \times \frac{-1+\sqrt{-3}}{2} + \sqrt[3]{q-\sqrt{(q^2-p^3)}} \times \frac{-1-\sqrt{-3}}{2}$.

(3.°) $x = \sqrt[3]{q+\sqrt{(q^2-p^3)}} \times \frac{-1-\sqrt{-3}}{2} + \sqrt[3]{q-\sqrt{(q^2-p^3)}} \times \frac{-1+\sqrt{-3}}{2}$.

Ubi

Ubi quantitatum imaginariarum sic ordo disponitur, quia quum antea factum fuerit $yz = p$, primi valoris termini ita sunt cum imaginariis conjungendi, ut mutua ipsarum y, z multiplicatio speciem p semper restituat, quod non accidit, si aliis modis præter expositos quantitates imaginariæ distribuantur.

587. *Scholion* II. Si quis interea cupiat formulas allatas ad examen revocare, ponat brevitatis gratia $\sqrt[3]{q + \sqrt{(q^2 - p^3)}} = m$; $\sqrt[3]{q - \sqrt{(q^2 - p^3)}} = n$. Ex facto secundæ formulæ $x + m \times \frac{1 - \sqrt{-3}}{2} + n \times \frac{1 + \sqrt{-3}}{2}$ in tertiam $x + m \times \frac{1 + \sqrt{-3}}{2} + n \times \frac{1 - \sqrt{-3}}{2}$ habetur $xx + mx + nx + mm - mn + nn$, quod per primam $x - m - n$ multiplicatum præbet $x^3 - 3mnx - m^3 - n^3$; restitutisque quantitatum m, n, valoribus, tandem provenit $x^3 - 3px - 2q = 0$.

588. Propositio II. *Examinare, an radices æquationis cubicæ secundo termino mulctatæ sint reales, aut imaginariæ.*

Resolutio. Quoniam si binomium cubicum terminis constet altero rationali, altero irrationali, etiam radix terminis altero rationali, irrationali altero

tero ex cuborum genesi constare debet, finge, cubi $q \pm \sqrt{(q^2 - p^3)}$ radicem esse $m \pm \sqrt{n}$; ergo

(1.°) $\sqrt[3]{q+\sqrt{(q^2-p^3)}} + \sqrt[3]{q-\sqrt{(q^2-p^3)}} = m+\sqrt{n}+m-\sqrt{n} = 2m.$

(2.°) $\sqrt[3]{q+\sqrt{(q^2-p^3)}} \times -\frac{1+\sqrt{-3}}{2} + \sqrt[3]{q-\sqrt{(q^2-p^3)}} \times -\frac{1-\sqrt{-3}}{2} = (m+\sqrt{n}) \times -\frac{1+\sqrt{-3}}{2} + (m-\sqrt{n}) \times -\frac{1-\sqrt{-3}}{2} = -m+\sqrt{-3n}.$

(3.°) $\sqrt[3]{q+\sqrt{(q^2-p^3)}} \times -\frac{1-\sqrt{-3}}{2} + \sqrt[3]{q-\sqrt{(q^2-p^3)}} \times -\frac{1+\sqrt{-3}}{2} = (m+\sqrt{n}) \times -\frac{1-\sqrt{-3}}{2} + (m-\sqrt{n}) \times -\frac{1+\sqrt{-3}}{2} = -m-\sqrt{-3n}.$

Tres igitur æquationum cubicarum radices sunt $2m$; $-m+\sqrt{-3n}$; $-m-\sqrt{-3n}$. Unde statim colligitur, primæ formulæ valorem semper esse realem, quamvis apparere ipsa possit sub aspectu imaginario, quod accidit cum $p^3 > q^2$; secundam vero,

ac

ac tertiam formulam ſemper ſufficere valorem realem cum n eſt quantitas negativa, ſeu cum $p^3 > q^2$, & contra valorem imaginarium cum n eſt quantitas poſitiva, ſeu cum $p^3 < q^2$. Quod ſi fuerit $q^2 = p^3$, ſeu $\sqrt{(q^2 - p^3)} = 0$, tunc duæ radices æquationis erunt æquales & omnes reales; quum enim in eo caſu ſit $\sqrt{-3n} = 0$, prima radix erit $2m$, altera $-m$, tertia $-m$, quarum ſumma $2m - 2m = 0$, ut fieri debet; *Quod erat* &c.

589. *Corollarium.* Ergo generatim cum in æquatione cubica ſecundo termino mutilata tertii termini coëfficiens eſt poſitivus, notum eſt ex demonſtratis (488.), æquationem duas continere radices imaginarias. Cum tertii termini coëfficiens eſt negativus, & $p^3 < q^2$, æquatio duas itidem continet radices imaginarias. Demum cum tertii termini coëfficiens eſt negativus, & $p^3 > q^2$, radices omnes ſunt reales & inæquales.

590. Propoſitio III. *Si æquationis cubicæ ſecundo termino carentis radices ſint omnes reales, & inæquales, prima formula generalis* $\sqrt[3]{q + \sqrt{(q^2 - p^3)}} + \sqrt[3]{q - \sqrt{(q^2 - p^3)}}$ *ſub aſpectu imaginario apparere ſemper debet.*

Dem. Quum ob radices omnes reales tertii termini coëfficiens ſit neceſſario negativus, æquatio iſta cubica erit $x^3 - 3px \pm 2q = 0$; quare poſita ut ſupra $x = y + \frac{p}{y}$, ſeu $xy - yy = p$, ſeu $yy - xy = -$

$-p$, elicitur $y = \frac{1}{2}n \pm \frac{1}{2}\sqrt{n^2 - 4p}$. Cum igitur fit $n < \sqrt{4p}$, conftat y, adeoque etiam $\frac{p}{y}$ evadere quantitatem imaginariam. Id autem evenit, quotiefcumque omnes æquationes radices funt reales & inæquales, quod fic demonftro.

Sint tres radices reales, & inæquales $a, -b, -c$, five $-a, +b, +c$, nimirum fint æquationes fimplices $x \mp a = 0$, $x \pm b = 0$, $x \pm c = 0$, & ob fecundi termini defectum fit $a = b + c$; æquatio cubica ex harum multiplicatione prodiens erit

$$\left.\begin{array}{l} x^3 - abx \mp abc \\ \quad - acx \\ \quad + bcx \end{array}\right\} = 0 \quad \text{feu valorem fpeciei } a \text{ fubftituendo} \quad \left\{\begin{array}{l} x^3 - b^2x \mp bc(b+c) \\ \quad - bcx \\ \quad - c^2x \end{array}\right\} = 0.$$

Jam vero $2bc < b^2 + c^2$ (74.), & $3bc < b^2 + c^2 + bc$; hinc $bc < \frac{1}{3}(b^2 + c^2 + bc)$; & addendo utrinque $b^2 + bc + c^2$, fiet $b^2 + 2bc + c^2 < \frac{4}{3}(b^2 + bc + c^2)$; & radicem extrahendo, $b + c < \pm \sqrt{\frac{4}{3}(b^2 + bc + c^2)}$; fed $\sqrt{\frac{4}{3}(b^2 + bc + c^2)} = \sqrt{\frac{4 \times 3p}{3}} = \sqrt{4p}$, & $b + c$

$b+c=a$; ergo $a<\pm\sqrt{4p}$; proptereaque reliquæ duæ radices, minores singillatim quam a, multo magis erunt minores quam $\pm\sqrt{4p}$; ergo in hac hypothesi erit semper $x<\sqrt{4p}$.

$$\text{Sed } y = \sqrt[3]{q \pm \sqrt{(q^2-p^3)}}, \; \& \; \frac{p}{y} =$$

$\sqrt[3]{q \mp \sqrt{(q^2-p^3)}}$ per paulo ante demonstrata; ergo &c. *Quod erat* &c.

591. *Scholion.* Casus iste, qui mathematicorum diu multumque torsit ingenia, non mirum, si ejus correctioni adlaborantium conatus semper eluserit; neque enim aliter (allatis manentibus conditionibus) nisi sub expressione imaginaria, dicta formula sese prodere potest, sic problematis natura exigente. Koënigius rem perpendens, ex æquationis

superioris $y=\frac{1}{2}x\pm\frac{1}{2}\sqrt{x^2-4p}$ aspectu arbitratus est, tunc incognitam x evadere imaginariam, cum sit $x^2<4p$, vel $x<\sqrt{4p}$, adeoque Cardani methodum hypothesim hoc in casu impossibilem involvere; miror autem Virum clarissimum non modo sese inconsiderato isto judicio decepisse, sed & alios in eundem errorem traxisse; quum enim ha-

beatur $x=y+\frac{p}{y}$, sive $x=y+z$, patet aggregatum istud duabus quantitatibus imaginariis constans æquivalere huic $A+B\sqrt{-1}+A-B\sqrt{-1}=2A$, quan-

quantitati nempe reali, ut etiam paulo ante (588.) vidimus.

In similem errorem lapsus est alius illustris Mathematicus, qui sic ratiocinatur.

Sint reales & inæquales radices æquationis $x^3 - 3px \pm 2q = 0$, in qua, ut supra, $x = y + z$, ipsarumque radicum maxima ponatur $= \pm 2a$, reliquæ vero $\mp a \mp b$, & $\mp a \pm b$, ita ut tres fiant æquationes simplices $x \pm 2a = 0$; $x \mp a \mp b = 0$; $x \mp a \pm b = 0$. Ex harum multiplicatione oritur æquatio $x^3 - (3a^2 - b^2)x \pm 2a^3 \mp 2ab^2 = 0$ ejusdem formæ ac proposita. Comparatis terminis homologis, elicitur $3a^2 + b^2 = 3p$, seu $a^2 + \frac{b^2}{3} = p$, & $a^2 = p - \frac{b^2}{3}$, quare $a^2 < p$, & terminus $3px$ signo semper afficitur negativo. Jam vero quoniam $yz = p$, positumque fuit $a^2 + \frac{b^2}{3} = p$, hoc est $a^2 = p - \frac{b^2}{3}$, stabit $a^2 < yz$, videlicet quadratum a^2 genitum a quadrato dimidiæ radicis $2a$, erit minus rectangolo prodeunti ex suis partibus inæqualibus y, z, quod est impossibile. Haud mirum igitur, si ab expressione impossibili expressiones semper impossibiles oriuntur.

Ingeniosum sane, sed fallax ratiocinium. Quum enim esse y, & z in hac hypothesi nequeant ex demonstratis nisi quantitates imaginariæ, eadem hic subito locum non obtinet conditio, quæ in quantitatibus realibus. Profecto ob $2a = x = y + z = A + B\sqrt{-1} + A - B\sqrt{-1} = 2A$, erit $a = A$,

& $a^2 = A^2$. Hinc $yz = p = (A + B\sqrt{-1})(A - B\sqrt{-1})$ $= A^2 + B^2 = a^2 + B^2$; quare $a^2 < A^2 + B^2$, sive $a^2 < p$. Sed $a^2 = \frac{x^2}{4}$; ergo $\frac{x^2}{4} < p$, ac tandem $x < \sqrt{4p}$; quod quum semper debeat accidere (590.), casus iste non est inter impossibiles, ut cel. Auctor autumat, sed inter necessarios referendus.

Methodo Cardanica a somniatæ imperfectionis accusationibus satis superque vindicata, ejusque veritate per secundam, quam attulimus, Problematis exactam resolutionem, nec non per examen (587.) extra dubitationis aleam posita, in æquationes cubicas oculos rursus convertamus, oportet. Itaque data quavis æquatione cubica, cui desit secundus terminus, statim innotescet, utrum habeat nec ne radices imaginarias. Si istis careat, per signorum consecutiones ac permutationes facile colligitur, utrum duas contineat radices reales negativas, & unam positivam duabus negativis æqualem, an contra. Si duas habeat radices æquales, per traditas methodos (517.) istæ invenientur. Si unam saltem habeat radicem rationalem, hæc sine formularum generalium subsidio ex dictis (493. 495.) investigari potest, tum reliquæ per æquationis depressionem obtinentur. Si nulla inveniri potest radix rationalis; vel omnes radices irrationales sunt reales & inæquales, vel unam realem irrationalem cum duabus imaginariis æquatio comprehendet. Primus casus vere *irreductibilis*, seu *irresolutus* vocari meretur, nullam æquationis algebricam resolutionem nisi

per

per approximationem admittens. Dixi *verè* irreducibilis, quia etiam casus ille, ubi radices sunt omnes reales, & inæquales, sed rationales, sic solet appellari, eo quod prima formula generalis, seu Cardanica, sub aspectu imaginario se prodit. At hoc in casu radices extractæ per demonstrata (554.) semper habentur, quod in altero, ubi nempe radices sunt omnes irrationales, fieri nequit. Jam vero ad formularum generalium supra expositarum usum ostendendum per sequentia transeamus

Exempla.

I. Sit $x^3 * - 9x + 28 = 0$. Quoniam $9^3 = 729 < 28^2 = 784$, certum est, æquationem duas radices imaginarias continere; quare si radix realis per ultimi termini divisores inquiratur, reperietur, esse $= -4$. At si per formulam Cardanicam postuletur, erit $-3p = -9$, $p = 3$; $-2q = 28$, $q = -14$; factisque substitutionibus, habetur $\sqrt[3]{q + \sqrt{(q^2 - p^3)}}$ $= \sqrt[3]{-14 + \sqrt{(196 - 27)}} = \sqrt[3]{(-14 + 13)}$ $= \sqrt[3]{-1} = -1$, quæ est prima pars radicis. Præterea $\sqrt[3]{q - \sqrt{(q^2 - p^3)}} = \sqrt[3]{-14 - \sqrt{169}} =$ $\sqrt[3]{-14 - 13} = \sqrt[3]{-27} = -3$, quæ est secunda pars radicis; quare $\sqrt[3]{q + \sqrt{(q^2 - p^3)}} + \sqrt[3]{q - \sqrt{(q^2 - p^3)}}$

$=$

$= -4$; quapropter si per $x+4$ æquatio data ad quadraticam deprimatur, atque inde radix extrahatur, reliquæ duæ radices imaginariæ reperientur, nimirum $2+\sqrt{-3}$, $2-\sqrt{-3}$, quarum tamen valor etiam deprehenditur, si secunda, & tertia formula generalis adhibeatur, nam $-1 \times -\frac{1+\sqrt{-3}}{2}$,

& $-3 \times -\frac{1-\sqrt{-3}}{2} = 2+\sqrt{-3}$, ac similiter

$-1 \times -\frac{1-\sqrt{-3}}{2}$, & $-3 \times -\frac{1+\sqrt{-3}}{2} = 2 -\sqrt{-3}$. Demum $2+\sqrt{-3}+2-\sqrt{-3}=4$; & $4-4=0$, ut secundi in æquatione termini absentia requirit.

II. Sit $x^3+6x-20=0$. Hæc etiam æquatio ob tertium terminum $6x$ positivum duas habet radices imaginarias (488.). Si radicem realem, cujus inventio per ultimi termini divisores est facillima, per primam formulam generalem indagare libeat, erit $-3p=6$, $p=-2$; $-2q=-20$, $q=10$; quare

$$x = \sqrt[3]{10+\sqrt{(100+8)}} + \sqrt[3]{10-\sqrt{(100+8)}}$$

$= \sqrt[3]{10+6\sqrt{3}} + \sqrt[3]{10-6\sqrt{3}}$. Itaque ex binomiis cubicis $10+6\sqrt{3}$, $10-6\sqrt{3}$ radices extractæ erunt $1+\sqrt{3}$, $1-\sqrt{3}$, quarum summa $=2=x$. Reliquæ duæ imaginariæ sunt $-1+3\sqrt{-1}$, $-1-3\sqrt{-1}$, quarum summa $=-2$, adeoque $2-2=0$, ut fieri oportebat.

III.

III. Sit $x^3 - 7x - 6 = 0$. Quia tertius terminus est negativus, & $7^3 = 343 > 6^2 = 36$, tres æquatio habet radices reales, duas nempe negativas, & unam positivam, quæ facile per ultimi termini divisores reperiuntur, adeoque rationales. Si igitur per primam formulam generalem una ex illis quæratur, sub expressionis imaginariæ forma repræsentabitur. Profecto quum sit $-3p = -7$; $p = \frac{7}{3}$, $-2q = -6$, $q = 3$, substituendo prodibit $\sqrt[3]{q + \sqrt{(q^2 - p^3)}}$

$$= \sqrt[3]{3 + \sqrt{(9 - \tfrac{343}{27})}}; = \sqrt[3]{3 + \sqrt{-\tfrac{100}{27}}};$$

quare $x = \sqrt[3]{3 + \sqrt{-\tfrac{100}{27}}} + \sqrt[3]{3 - \sqrt{-\tfrac{100}{27}}}$.

Itaque si ex binomio $3 \pm \sqrt{-\frac{100}{27}}$ radix cubica extrahatur, emerget $-1 \pm \sqrt{-\frac{4}{3}}$ (547. 554.). Hinc $x = -1 + \sqrt{-\frac{4}{3}} - 1 - \sqrt{-\frac{4}{3}} = -2$, quæ est una ex propositæ radicibus. Si quis cupiat reliquas radices -1, & 3 ex formulis generalibus secunda & tertia depromere, is post radicis ex binomio cubico extractionem voti compos nullo negotio fiet.

Sed aliis etiam modis diversis ab expositis eundem finem adsequi possumus, quum enim quilibet cubus tres habeat radices, tres etiam radices ex bi-

nomio cubico $3+\sqrt{-\frac{100}{27}}$ elicientur, quarum una $-1+\sqrt{-\frac{4}{3}}$ inventa, altera erit $(-1+\sqrt{-\frac{4}{3}})\times-\frac{1+\sqrt{-3}}{2}$, tertia $(-1+\sqrt{-\frac{4}{3}})\times-\frac{1-\sqrt{-3}}{2}$, five conjunctim $-1+\sqrt{-\frac{4}{3}}$; $-\frac{1}{2}-5\sqrt{-\frac{1}{12}}$; $\frac{3}{2}+\sqrt{-\frac{1}{12}}$.

Ergo fi pro radice cubica binomii partim rationalis, partim irrationalis ponatur ut fupra (588.) indeterminate $m+\sqrt{n}$, fiatque cum prima binomii cubici radice $-1+\sqrt{-\frac{4}{3}}$ modo extracta comparatio, erit $m=-1$, $n=-\frac{4}{3}$; qui valores fi fubftituantur in formulis fimplicioribus radices omnes cubicas repræfentantibus, videlicet $2m$; $-m+\sqrt{-3n}$; $-m-\sqrt{-3n}$; habetur $2m=-2$; $-m+\sqrt{-3n}=3$; $-m-\sqrt{-3n}=-1$. Itidem fi ejufdem binomii cubici $3+\sqrt{-\frac{100}{27}}$ adfumatur radix altera $-\frac{1}{2}-5\sqrt{-\frac{1}{12}}$, erit $m=-\frac{1}{2}$; $n=$

—

$-\frac{25}{12}$; idcirco $2m = -1$; $-m + \sqrt{-3n} = 3$; $-m - \sqrt{-3n} = -2$. Accepta denique tertia ipsius binomii radice $\frac{3}{2} + \sqrt{-\frac{1}{12}}$, fiet $m = \frac{3}{2}$; $n = -\frac{1}{12}$; proinde $2m = 3$; $-m + \sqrt{-3n} = -1$; $-m - \sqrt{-3n} = -2$. Ergo unaquæque ex tribus dicti binomii cubici radicibus æquationis propositæ tribus radicibus eruendis æque inservit. Quin etiam si binomii cubici negativi $3 - \sqrt{-\frac{100}{27}}$ in usum vocetur radix $-1 - \sqrt{-\frac{4}{3}}$, ac reliquæ duæ ut supra educantur, quæ erunt $\frac{3}{2} - \sqrt{-\frac{1}{12}}$, & $-\frac{1}{2} + 5\sqrt{-\frac{1}{12}}$, fiantque singillatim tres solitæ substitutiones, tres eadem radices -1, -2, 3 rursus emergent. Sex igitur modis per triplicem binomii cubici cum affirmativi, tum negativi radicem tres obtinere possumus æquationis cubicæ propositæ radices; quod non erat omittendum, id enim in qualibet hujusmodi æquatione omnes radices reales, & rationales habente debet accidere.

592. Propositio IV. *Æquationem cubicam* casum irreductibilem *involventem per approximationem, sinuum ope, resolvere.*

Resolutio. In formula $\sin 3A = 3sc^2 - s^3$ (224.) $= 3rrs - 4s^3$ (ob $c^2 = r^2 - s^2$), exprimit s sinum A æqualem tertiæ parti sinus $3A$; sed si detur arcus a, cujus sinus $= b$, quum tertia pars arcus a sit incognita, etiam ejus sinus erit incognitus. Fiat igitur hoc in casu $s = x$; erit $\sin a = brr$ (336.) $= 3rrx - 4x^3$, seu $x^3 - \frac{3}{4} rrx + \frac{1}{4} brr = 0$, quæ æquatio duas habet radices veras, & unam falsam. In mentem nunc veniat, quod si omnes arcus possibiles eodem sinu b gaudentes trifariam secentur, sinus ad hujusmodi arcus pertinentes esse nequeunt nec plus nec minus quam tres (217.), qui reduci possunt ad sequentes, nempe $\sin \frac{1}{3} a$; $\sin \frac{180° - a}{3}$; $- \sin \frac{180° + a}{3}$, duos scilicet affirmativos, & unum negativum, ut exigit æquatio, seu $\sin \frac{1}{3} a$; $\sin (60° - \frac{1}{3} a)$; $- \sin (60° + \frac{1}{3} a)$.

Hisce stantibus, resolvenda proponatur æquatio $x^3 - 3px + 2q = 0$. Comparatis cum æquatione

$x^3 -$

$x^3 - \frac{3}{4} rrx + \frac{1}{4} brr = 0$ terminis homologis, elicitur $\frac{3}{4} r^2 = 3p$ seu $r = \sqrt{4p} = 2\sqrt{p}$; $\frac{1}{4} brr = 2q$, seu $b = \frac{8q}{rr} = \frac{2q}{p}$ ob $\frac{4}{rr} = \frac{1}{p}$. Radio igitur $= 2\sqrt{p}$ describatur circulus, cujus arcus habens sinum $= \frac{2q}{p}$ fiat $= a$; erunt $\sin \frac{1}{3} a$; $\sin (60^\circ - \frac{1}{3} a)$; $- \sin (60^\circ + \frac{1}{3} a)$ tres incognitæ x valores. Sed sinus est minor radio; ergo $2\sqrt{p} > \frac{2q}{p}$, seu $p > \frac{q^2}{p^2}$, seu $p^3 > q^2$, qui est casus irreductibilis in æquationibus cubicis; adeoque omnes hujusmodi æquationes ad hunc casum relatæ per hanc methodum resolvi possunt.

Ut autem calculus ad Tabularum vulgarium sinus accommodetur, radius per Tabulas expressus fiat $= R$. Ex trigonometricis stabit $r : b = R : \sin a = \frac{Rb}{r} = R \times \frac{2q}{pr} = R \times \frac{q}{p\sqrt{p}}$. Habemus igitur per Tabulas valorem sinus ad arcum datum a per-

pertinentis, quare cognita a, innotescent etiam $\sin \frac{1}{3} a$; $\sin(60° - \frac{1}{3} a)$; $-\sin(60° + \frac{1}{3} a)$; itaque ad radices æquationis ex Tabulis eruendas fiat $R : r = \sin \frac{1}{3} a : x$; una radix erit $x = \frac{r \times \sin \frac{1}{3} a}{R} = \frac{2\sqrt{p}}{R} \times \sin \frac{1}{3} a$. Eodem modo reliquæ duæ radices invenientur, nimirum $x = \frac{2\sqrt{p}}{R} \times \sin(60° - \frac{1}{3} a)$, & $x = -\frac{2\sqrt{p}}{R} \times \sin(60° + \frac{1}{3} a)$. Ergo factis in casu particulari substitutionibus, æquatio resolvetur. *Quod erat* &c.

Exempla.

I. Sit $x^3 - 3x + 1 = 0$. Ergo $3p = 3$, $p = 1$; $2q = 1$, $q = \frac{1}{2}$; $R \times \frac{q}{p\sqrt{p}} = \frac{1}{2} R = \sin a$, quare $a = 30°$, adeoque tres incognitæ x valores sunt $x = \frac{2 \sin 10°}{R} = 0,3472964$; $x = \frac{2 \sin 50°}{R} = 1.$

$= 1.5320888$; $x = -\frac{2 \sin 70^\circ}{R} = -1{,}8793852$.

II. Sit $x^3 - x + \frac{1}{3} = 0$. Ergo $3p = 1$, $p = \frac{1}{3}$; $2q = \frac{1}{3}$, $q = \frac{1}{6}$; $R \times \frac{q}{p\sqrt{p}} = \frac{R}{2\sqrt{\frac{1}{3}}} = \frac{1}{2} R\sqrt{3} = \sin a$. Hinc quoniam $\frac{1}{2} R\sqrt{3}$ exprimit semilatus trianguli æquilateri circulo inscripti, erit $a = 60^\circ$, ac proinde tres æquationis radices sunt $x = \frac{2 \sin 20^\circ}{R\sqrt{3}} = 0{,}394931$; $x = \frac{2 \sin 40^\circ}{R\sqrt{3}} = 0{,}742227$; $x = \frac{-2 \sin 80^\circ}{R\sqrt{3}} = -1{,}137158$.

III. Sit $x^3 - 5x + 3 = 0$. Ergo $3p = 5$, $p = \frac{5}{3}$; $2q = 3$, $q = \frac{3}{2}$; $R \times \frac{q}{p\sqrt{p}} = R \times \frac{3}{2} : \frac{5\sqrt{5}}{3\sqrt{3}} = \frac{1}{2} R \sqrt{\frac{3^5}{5^3}} = \sin a$. Sumptisque logarithmis, habetur $l \sin a = lR + \frac{5}{2} l3 - \frac{3}{2} l5 - l2$; ergo

go $a = 44^\circ 11' 52''$, & tres radices sunt $x =$

$$\frac{2\sqrt{5}\, \mathit{fin}\, (14^\circ 43' 57'')}{R\sqrt{3}};\ x = \frac{2\sqrt{5}\ \mathit{fin}\ (45^\circ 16' 3'')}{R\sqrt{3}};$$

$x = \frac{-2\sqrt{5}\, \mathit{fin}\, (74^\circ 43' 57'')}{R\sqrt{3}}$; atque inito calculo invenietur $x = 0,656625$; $x = 1,834238$; $x = -2,490863$.

593. *Monitum.* Si æquatio data sit $x^3 - px - q = 0$, reducetur ad formam præcedentem $x^3 - px + q = 0$, faciendo $x = -y$; vel, quod eodem recidit, signa in æquationis $x^3 - px + q = 0$ radicibus inventis mutentur (443.).

CAPUT IX.

DE ÆQUATIONIBUS BIQUADRATICIS, EARUMQUE RESOLUTIONE.

594. Egimus jam de æquationibus biquadraticis puris (474. 480.), quare scimus, æquationem $x^4 + p = 0$ omnes habere radices imaginarias, & $x^4 - p = 0$ duas radices reales $+\sqrt[4]{p}$, $-\sqrt[4]{p}$ continere, reliquis existentibus imaginariis. Quoad æquationes biquadraticas derivativas sit sequens

595. Propositio I. *Quot habeat data æquatio biquadratica derivativa radices cum reales, tum imaginarias statim detegere.*

Re-

Resolutio. Sit $x^4+px^2+q=0$. Ponatur $x^2=z$; fiet transformata $z^2+pz+q=0$. Hujus radices problema resolvent; si enim duas habeat transformata radices reales & positivas, quatuor æquationis propositæ radices erunt itidem reales, sed duæ positivæ, & duæ negativæ. Si vero manentibus realibus transformatæ radicibus, altera sit positiva, altera negativa, duæ æquationis propositæ radices erunt imaginariæ, ac duæ reliquæ reales, sed una positiva, altera negativa. Si denique duæ transformatæ radices sint negativæ aut imaginariæ, quatuor ejusdem propositæ radices erunt omnes imaginariæ. Hæc ex formularum jam traditarum (309.) inspectione facile derivantur. In primo enim casu habemus $x=\pm\sqrt[4]{p\pm\sqrt{(p^2-q)}}$ quia $p^2>q$; in secundo $x=\pm\sqrt[4]{-p\pm\sqrt{(p^2+q)}}$, vel $x=\pm\sqrt[4]{p\pm\sqrt{(p^2+q)}}$; in tertio $x=\pm\sqrt[4]{-p\pm\sqrt{(p^2-q)}}$. *Quod erat* &c.

596. *Monitum.* Ut ad æquationes affectas quarti gradus transeamus, istæ, quas secundo termino semper expertes adsumimus, sub hac expressione generali $x^4 * \pm px^2 \pm qx \pm r=0$ comprehenduntur, unde octo oriuntur formulæ sequentes

I.

I. $x^4 + px^2 + qx + r = 0$
II. $x^4 + px^2 + qx - r = 0$
III. $x^4 + px^2 - qx - r = 0$
IV. $x^4 + px^2 - qx + r = 0$
V. $x^4 - px^2 + qx + r = 0$
VI. $x^4 - px^2 - qx + r = 0$
VII. $x^4 - px^2 - qx - r = 0$
VIII. $x^4 - px^2 + qx - r = 0$

quarum primæ quatuor duas radices imaginarias continent, adeoque omnes æquationes, quæ ad ipsas referuntur, duabus saltem radicibus imaginariis constabunt.

397. Propositio II. *Omnis æquatio biquadratica secundo termino carens in duo trinomia quadratica realia resolvitur.*

Dem. Esto $x^4 + px^2 + qx + r = 0$, sive $x^4 + px^2 = -qx - r$. Adde hinc inde mx^2; fiet $x^4 + px^2 + mx^2 = mx^2 - qx - r$, seu $x^4 + (p+m)x^2 = m\left(x^2 - \frac{qx}{m} - \frac{r}{m}\right)$. Ut primum membrum evadat quadratum perfectum (305. n. III.), ex utraque parte adde $\frac{(p+m)^2}{4}$; habebis $x^4 + (p+m)x^2 + \frac{(p+m)^2}{4} = m\left(x^2 - \frac{qx}{m} - \frac{r}{m} + \frac{(p+m)^2}{4m}\right)$; & extracta radice quadrata, prodit $x^2 + \frac{p+m}{2} = \sqrt{m} \times \sqrt{x^2 - \frac{qx}{m} - \frac{r}{m} + \frac{(p+m)^2}{4m}}$. Ut autem ex secundo mem-

membro (omisso radicali $\sqrt{m}$) fiat quadratum perfectum, esse ex jam dictis (65. n. 2.) debet $\frac{q^2x^2}{4m^2} = x^2 \times \left(-\frac{r}{m} + \frac{(p+m)^2}{4m}\right)$, sive $\frac{q^2}{4m^2} = -\frac{r}{m} + \frac{(p+m)^2}{4m}$. Substitue hunc valorem, & reperies $x^2 + \frac{p+m}{2} = \sqrt{m} \times \sqrt{x^2 - \frac{qx}{m} + \frac{q^2}{4m^2}}$, vel $x^2 + \frac{p+m}{2} = \sqrt{m} \times \left(\pm x \mp \frac{q}{2m}\right)$, vel $x^2 \mp x\sqrt{m} \pm \frac{q}{2\sqrt{m}} + \frac{p+m}{2} = 0$. Ista æquatio in duo trinomia $x^2 - x\sqrt{m} + \frac{q}{2\sqrt{m}} + \frac{p+m}{2} = 0$ $x^2 + x\sqrt{m} - \frac{q}{2\sqrt{m}} + \frac{p+m}{2}$ (secundos terminos signis diversis affectos habentia, ut secundi termini defectus exigit) resolvitur; quibus invicem multiplicatis, & pro $\frac{q^2}{4m^2}$ subrogato in producto ejus valore

$-\frac{r}{m}+\frac{(p+m)^2}{4m}$, reviviſcet æquatio $x^4+(p+m)x^2$ $=mx^2-qx-r=0$. *Quod erat primum.*

Ut m determinetur, multiplico $\frac{q^2}{4m^2}=-\frac{r}{m}+\frac{(p+m)^2}{4m}$ per m, & obtineo $q^2=-4rm+m(p+m)^2$; unde elicio $m^3+2pm^2+p^2m-q^2=0$
$\qquad\qquad\qquad\qquad +\ p^2-4rm$
quæ tertii gradus æquatio quum unam ſaltem habeat neceſſario radicem realem (424.), demonſtrat, valorem quantitatis m eſſe realem. Præterea quia dicta æquatio habet ultimum terminum q^2 negativum, quicumque ſit quantitatis q valor, poſitivus nempe, aut negativus, ejus quadratum q^2 ſit ſemper poſitivum; ac proinde unam ſaltem ipſa habebit radicem poſitivam. Ergo duo iſta trinomia ſemper erunt realia, quæcumque ſint æquationis biquadraticæ propoſitæ radices, reales nempe, aut imaginariæ. *Quod erat ſecundum.*

598. *Coroll.* 1. Omnis ergo æquatio quarti gradus $x^4+px^3+qx^2+rx+s=0$ reſolvi poteſt in duos factores reales ſecundi gradus, etiamſi factores omnes ſimplices habeat imaginarios.

599. *Coroll.* 2. Æquationes omnes derivativæ hujus formæ $x^{4m}+px^{3m}+qx^{2m}+rx^{m}+s=0$ in duo tri-

trinomia resolvi possunt; si enim ponatur $y = x^m$, proposita evadit $y^4 + py^3 + qy^2 + ry + s = 0$.

600. *Coroll.* 3. Quum æquatio quælibet quinti gradus unam habeat necessario radicem realem, unum factorem habebit realem, adeoque alter factor erit biquadraticus, sed iste in duos factores quadraticos reales resolvitur; ergo æquatio quælibet quinti gradus erit in tres factores reales resolubilis, quorum unus est simplex, & reliqui duo duplices, seu secundi gradus.

601. *Coroll.* 4. Quod si expressio cujuscumque gradus quatuor tantum habeat factores simplices imaginarios, quum eorum productum sit reale, & sub hac forma $x^4 + Ax^3 + Bx^2 + Cx + D = 0$; productum illud erit resolubile in duos factores quadraticos reales, quorum uterque duos factores simplices continebit.

Nonnulli dari possunt casus, qui allatam demonstrationem destruere videntur. Ne hac falsa veri specie decipiamur sit sequens.

602. Prop. III. *Proponatur* $x^4 + ax^3 + (b + 2)xx + ax + 1 = 0$; *Dico hanc æquationem in duos factores reales secundi gradus esse resolubilem.*

Dem. Ponamus, duos istos factores esse hujus formæ $(xx + px + 1)(xx + qx + 1) = 0 = x^4 + (p + q)x^3 + (pq + 2)x^2 + (p + q)x + 1$.

Conlatis terminis homologis (478.), erit $p + q = a$; $pq + 2 = b + 2$, seu $pq = b$. Propter $(p - q)^2 = p^2 + q^2 - 2pq = p^2 + q^2 + 2pq - 4pq$, erit $(p - q)^2$

Pars alt. Q =

$= a^2 - 4b$, & $p - q = \sqrt{a^2 - 4b}$. Hinc $p = \frac{a + \sqrt{a^2 - 4b}}{2}$, $q = \frac{a - \sqrt{a^2 - 4b}}{2}$, ſummando enim, redit $p - q = \sqrt{a^2 - 4b}$.

Quotieſcumque igitur $aa > 4b$, patet, factores memoratos eſſe reales, adeoque nulla hic occurrit dubitatio; at ſi $aa < 4b$, ipſi erunt imaginarii, quod demonſtratis averſatur. Ut hæc difficultas evaneſcat, duorum factorum quadraticorum, quos ſtatuimus, factores ſimplices intueamur qui ſunt

I. $x + \frac{1}{2}p + \frac{1}{2}\sqrt{pp - 4}$;

II. $x + \frac{1}{2}p - \frac{1}{2}\sqrt{pp - 4}$;

III. $x + \frac{1}{2}q + \frac{1}{2}\sqrt{qq - 4}$;

IV. $x + \frac{1}{2}q - \frac{1}{2}\sqrt{qq - 4} = 0$.

Quoniam vero $aa < 4b$, ponamus $4b = aa + cc$. Fiet $p = \frac{a + c\sqrt{-1}}{2}$, $q = \frac{a - c\sqrt{-1}}{2}$; quare

$\surd$

$$\left.\begin{array}{l}\sqrt{pp-4}=\frac{1}{2}\sqrt{(aa-cc-16+2ac\sqrt{-1}}\\ \\ \sqrt{qq-4}=\frac{1}{2}\sqrt{(aa-cc-16-2ac\sqrt{-1}}\end{array}\right\}$$

Quũ itaque ambæ iſtæ formulę ſint imaginariæ, ſtatuatur

$$\frac{1}{2}\sqrt{(aa-cc-16+2ac\sqrt{-1})}=u+v\sqrt{-1}$$

$$\frac{1}{2}\sqrt{(aa-cc-16-2ac\sqrt{-1})}=u-v\sqrt{-1}$$

Ergo valores ſubſtituendo, quatuor factores ſimplices evadent

I. $x+\frac{1}{4}(a+c\sqrt{-1})+\frac{1}{4}(u+v\sqrt{-1})=x+\frac{1}{4}(a+u)+\frac{1}{4}(c+v)\sqrt{-1}$

II. $x+\frac{1}{4}(a-c\sqrt{-1})-\frac{1}{4}(u+v\sqrt{-1})=x+\frac{1}{4}(a-u)+\frac{1}{4}(c-v)\sqrt{-1}$

III. $x+\frac{1}{4}(a+c\sqrt{-1})+\frac{1}{4}(u-v\sqrt{-1})=x+\frac{1}{4}(a+u)-\frac{1}{4}(c+v)\sqrt{-1}$

IV. $x+\frac{1}{4}(a-c\sqrt{-1})-\frac{1}{4}(u-v\sqrt{-1})=x+\frac{1}{4}(a-u)-\frac{1}{4}(c-v)\sqrt{-1}$

Duc I. in III. nec non II. in IV. & duo prodibunt factores quadratici reales

$$xx + \frac{1}{2}(a+u)x + \frac{1}{16}(aa+cc) + \frac{1}{16}(uu+vv) + \frac{1}{8}(au+cv)$$

$$xx + \frac{1}{2}(a-u)x + \frac{1}{16}(aa+cc) + \frac{1}{16}(uu+vv) - \frac{1}{8}(au+cv).$$ *Quod erat* &c.

603. *Scholion.* Ut duo postremi factores quadratici reales per æquationis propositæ literas habeantur, revocemus æquationes

$$\sqrt{(aa-cc-16+2ac\sqrt{-1})} = u + v\sqrt{-1}$$
$$\sqrt{(aa-cc-16-2ac\sqrt{-1})} = u - v\sqrt{-1}$$

unde quadrando, separando, & comparando singillatim quantitates cum reales tum imaginarias (436.), colligitur $uu - vv = aa - cc - 16$, & $uv = ac$. Sed $(uu-vv)^2 = (uu+vv)^2 - 4uuvv = (uu+vv)^2 - 4aacc$; ergo $(uu+vv)^2 = (uu-vv)^2 + 4aacc = (aa-cc-16)^2 + 4aacc = (aa+cc)^2 - 32(aa-cc) + 256$, & $uu+vv = \sqrt{(aa-cc)^2 - 32(aa-cc) + 256}$.
Hinc ob $uu + vv + (uu - vv) = 2uu$, habemus

$$u = \sqrt{\frac{\sqrt{(aa+cc)^2 - 32(aa-cc) + 256} + aa - cc - 16}{2}};$$

&

& ob $uu+vv-(uu-vv)=2vv$, prodit

$$v=\sqrt{\frac{\sqrt{(aa+cc)^2-32(aa-cc)+256}-aa+cc+16}{2}};$$

Itaque pro cc substituendo suum valorem $4b-aa$, fiet

$$u=\sqrt{\sqrt{4bb-16(aa-2b)+64}-8-2b+aa}$$

$$v=\sqrt{\sqrt{4bb-16(aa-2b)+64}+8+2b-aa}$$

sive

$$u=\sqrt{2\sqrt{(b+4)^2-4aa}-8-2b+aa}$$

$$v=\sqrt{2\sqrt{(b+4)^2-4aa}+8+2b-aa}$$

Ut autem formulæ $au+cv$ valor commodius exprimatur in quadrati $aauu+ccvv+2acuv$ valorem inquirentes, inveniemus

$aauu=2aa\sqrt{(b+4)^2-4aa}-8aa-2aab+a^4$;
$ccvv=\left.\begin{matrix}+8b\\-2aa\end{matrix}\right\}\sqrt{(b+4)^2-4aa}+32b+8bb-4aab$
$-8aa-2aab+a^4$; $2acuv=2aacc=8aab-2a^4$; quapropter stabit $(au+cv)^2=aauu+ccvv+2acuv=$

$8b\sqrt{(b+4)^2-4aa}+32b-16aa+8bb$; & radicem extrahendo,

$$au+cv=2\sqrt{2bb+8b-4aa+2b\sqrt{(b+4)^2-4aa}};$$

Ergo si $4b>aa$, duo factores reales æquationis datæ

 x^4+

$x^4 + ax^3 + (b+2)x^2 + ax + 1 = 0$ erunt $x^2 + \frac{1}{2}ax +$

$\frac{1}{2}x\sqrt{2\sqrt{(b+4)^2-4aa}-8-2b+aa} \quad +$

$\frac{1}{4}b + \frac{1}{4}\sqrt{(b+4)^2-4aa} \quad +$

$\frac{1}{4}\sqrt{2bb+8b-4aa+2b\sqrt{(b+4)^2-4aa}}\,;$

$x^2 + \frac{1}{2}ax - \frac{1}{2}x\sqrt{2\sqrt{(b+4)^2-4aa}-8-2b+aa}$

$+\frac{1}{4}b + \frac{1}{4}\sqrt{(b+4)^2-4aa} \quad -$

$\frac{1}{4}\sqrt{2bb+8b-4aa+2b\sqrt{(b+4)^2-4aa}}\,.$

604. Propositio IV. *Æquationes biquadraticas ad cubicas reducere.*

Resolutio. Sit $x^4 + px^2 + qx + r = 0$ æquationes biquadraticas quascumque (radicalibus, fractionibus, & secundo termino, si adsint, sublatis) repræsentans. Quoniam ista ex duarum quadraticarum multiplicatione consurgit (597.), finge duas hujusmodi æquationes componentes esse

$\left.\begin{aligned} xx + fx + g &= 0 \\ xx - fx + h &= 0 \end{aligned}\right\}$ quibus invicem ductis, emergit

$x^4 -$

$$\begin{array}{l} x^4 - ffxx - fgx + gh = 0 \\ \quad + g \quad + fh \\ \quad + h \end{array}$$

Quum itaque postrema hæc æquatio eadem sit (*ex hyp.*) ac proposita, terminis homologis in utraque comparatis erit (1.°) $-ff + g + h = p$; (2.°) $-fg + fh = q$; (3.°) $gh = r$; quare $g + h = p + ff$, seu facta per f multiplicatione, $fg + fh = fp + f^3$; At erat $-fg + fh = q$; ergo prius addendo, dein subtrahendo, duæ orientur æquationes

$$\left.\begin{array}{l} 2fh = f^3 + fp + q \\ 2fg = f^3 + fp - q \end{array}\right\} \text{quibus invicem multiplicatis, fit}$$

$4ffgh = f^6 + 2f^4p + ffpp - qq$; sed $gh = r$; ergo $4ffr = f^6 + 2pf^4 + ppf^2 - qq$; sive

$$\begin{array}{l} f^6 + 2pf^4 + ppf^2 - qq = 0 \\ \qquad\qquad -4r \end{array}$$

quæ est æquatio cubica derivativa; si enim ponatur $ff = z$, provenit *depressa*

$$\begin{array}{l} z^3 + 2pz^2 + ppz - qq = 0 \\ \qquad\qquad -4r \end{array}$$

; *Quod erat* &c.

605. *Coroll.* 1. Data igitur æquatione biquadratica speciali, facile habetur ejusdem cubica derivativa, substituendo in cubica derivativa generali pro p, q, r, valores respondentes datæ æquationis specialis propositæ. Sic si proponatur æquatio $x^4 - 7x^2 + 9x + 16 = 0$, erit $p = -7$, $q = 9$, $r = 16$; quibus valoribus in cubica derivativa generali suffectis, prodit cubica derivativa specialis

$$\begin{array}{l} f^6 - 14f^4 + 49f^2 - 81 = 0, \text{ seu } f^6 - 14f^4 - 15f^2 - 81 = 0. \\ \qquad\qquad -64 \end{array}$$

606. *Coroll.* 2. Si ex duabus æquationibus quadraticis componentibus $x^2 + fx + g = 0$, $x^2 - fx + h = 0$ extrahantur radices, invenientur esse

$$(1.) + \sqrt{(\tfrac{1}{4}ff - g)} - \tfrac{1}{2}f$$

$$(2.) - \sqrt{(\tfrac{1}{4}ff - g)} - \tfrac{1}{2}f$$

$$(3.) + \sqrt{(\tfrac{1}{4}ff - h)} + \tfrac{1}{2}f$$

$$(4.) - \sqrt{(\tfrac{1}{4}ff - h)} + \tfrac{1}{2}f$$

quæ sunt quatuor radices æquationis generalis propositæ, quarumque omnium summa æquatur zero, ut ob secundi termini defectum fieri oportebat.

607. *Coroll.* 3. Invento per æquationum cubicarum traditam resolutionem valore f, patet, reliquos quantitatum g, h valores facillime obtineri posse; etenim quia ex antecedenti calculo habuimus $2fg = f^3 + fp - q$; $2fh = f^3 + fp + q$; erit

$$g = \frac{f^3 + fp - q}{2f} = \frac{1}{2}f^2 + \frac{1}{2}p - \frac{1}{2}\frac{q}{f}$$

$$h = \frac{f^3 + fp + q}{2f} = \frac{1}{2}f^2 + \frac{1}{2}p + \frac{1}{2}\frac{q}{f}$$

quapropter si quantitas f prodeat rationalis, erunt quoque g, h rationales, & contra.

608. *Coroll.* 4. Quum cubica quævis derivativa sex habeat radices, sex erunt valores quantitatis f, adeoque sex etiam valores quantitatum g, h erui poterunt.

609.

609. *Scholion.* Quoniam æquatio derivativa generalis $f^6 - 2pf^4 - ppf^2 - qq = 0$ evadit $z^3 - 2pz^2$
$ - 4r$
$- ppz - qq = 0$, ex qua ſecundus terminus tolli
$- 4r$
debet, faciendo $z = y + \frac{2}{3}p$ (479.); ut operatio contrahatur, ob $z = ff$, pone $ff = y + \frac{2}{3}p$, & eadem æquatio ſecundo termino deſtituta obtinebitur. Generatim ſi proponatur æquatio $f^{3m} + af^{2m} + bf^m + c = 0$, hæc ſtatim reducetur ad æquationem tertii gradus ſecundo termino expertem, ſi fiat $f^m = y - \frac{1}{3}a$; adeoque omnis æquatio quatuor terminorum ſub hac forma $f^{3m} + af^{2m} + bf^m + c = 0$ recipit per Caput antecedens reſolutionem.

610. Propoſitio V. *Æquationes biquadraticas affectas reſolvere.*

Reſolutio. Sublato ſecundo termino (479.), æquatio biquadratica reducatur ad cubicam (605.). Inveſtigetur per aliquam ex allatis methodis (493. 495. 585.) ejus radix realis, factiſque per hanc ſubſtitutionibus, quatuor valores ſpeciales radicum generalium ſupra expoſitarum (606.), queis æquatio conſtat, eruentur. Seu quoniam ultimus æquationis derivativæ generalis terminus qq eſt ſemper quadratum, quod numeros quadratos pro diviſoribus habere

bere poteſt; niſi ſecundum terminum tollere lubeat, æquatio derivativa ſpecialis dividatur per æquationem ſimplicem, cujus alter terminus unus ſit ex dicti ultimi termini diviſoribus quadratis, ſique exacta prodeat diviſio, radices expetitæ facilius reperientur, ut ſequentia docent

Exempla.

I. Sit $x^4 - 17x^2 - 20x - 6 = 0$. Inſtituta cum formula generali $x^4 + px^2 + qx + r = 0$ terminorum homologorum comparatione, erit $p = -17, q = -20, r = -6$; quare cubica derivativa ſpecialis fiet $f^6 - 34f^4 + 313f^2 - 400 = 0$. Tollatur ſecundus terminus, ut valor f obtineatur; vel brevius dividatur ipſa æquatio cubica derivativa per $ff - 16$, quod 16 eſt unum ex diviſoribus quadratis ultimi termini 400; quum autem diviſio bene cedat (prodit enim quotus $f^4 - 18f^2 + 25 = 0$), erit $ff = 16$, & ſex radices æquationis cubicæ derivativæ reperientur

$$4; -4; \sqrt{9 + 2\sqrt{14}}; -\sqrt{9 + 2\sqrt{14}}; \sqrt{9 - 2\sqrt{14}};$$

$-\sqrt{9 - 2\sqrt{14}}$; quarum primæ duæ rationales, nimirum $+4$, & -4, in quatuor memoratas radices (607.) introductæ una cum repertis quantitatum g, h valoribus, quatuor exhibebunt æquationis propoſitæ radices, nempe $-2 + \sqrt{2}$; $-2 - \sqrt{2}$; $2 + \sqrt{7}$; $2 - \sqrt{7}$. Etenim ſi ſumatur $f = 4$, erit $g = 2$, $h = -3$; ſi fiat $f = -4$, erit $g = -3$; $h = 2$. Quatuor reliquæ radices æquationis cubicæ derivati-

væ

væ non ſuppeditant radices æquationis propoſitæ, niſi ſub forma irreductibili, adeoque inutiles.

611. *Coroll.* 1. Ergo ſi æquatio biquadratica propoſita omnes habeat radices reales, etiam ſua cubica derivativa depreſſa tres habebit radices reales. Si enim ponatur $ff = z$, derivativa depreſſa evadet $z^3 - 34z^2 + 313z - 400 = 0$, cujus tres radices 16; $9 + 2\sqrt{14}$; $9 - 2\sqrt{14}$ ſunt omnes reales.

612. *Coroll.* 2. Hinc ſi cubica derivativa omnes haberet radices reales irrationales, in caſum irreſolutum incideremus.

II. Sit $x^4 - 86x^2 + 600x - 851 = 0$. Ergo $p = -86$; $q = 600$; $r = -851$; ſubſtitutiſque his valoribus in cubica derivativa generali $f^6 + 2pf^4 + ppf^2 - qq = 0$, orietur ſpecialis $f^6 - 172f^4 + \atop - 4r$ $10800f^2 - 360000 = 0$; quam ſi dividas per $ff - 100 = 0$, id fit ſine reſiduo, ac proinde $ff = 100$, & $f = \pm 10$; hinc ſi ſumatur $f = 10$, erit $g = -23$, $h = 37$; ſi $f = -10$, erit $g = 37$, $h = -23$; Ergo radices quæſitæ $-5 + \sqrt{48}$; $-5 - \sqrt{48}$; $+5 + \sqrt{-12}$; $+5 - \sqrt{-12}$; duæ nimirum reales, ac duæ imaginariæ.

613. *Coroll.* Ex diviſione cubicæ derivativæ $f^6 - 172f^4 + 10800f^2 - 360000 = 0$ per $ff - 100 = 0$ prodit $f^4 - 72f^2 + 3600 = 0$; ex quo $f^2 = 36 \pm \sqrt{-2304}$, & $f = \pm \sqrt{36 \pm \sqrt{-2304}}$. Ergo ſi æquatio biquadratica propoſita duas habeat radices

ces reales, & duas imaginarias, duæ radices imaginariæ erunt quoque in cubica depressa, quæ hic est $z^3 - 172z^2 + 10800z - 360000 = 0$, ubi $z = ff = 36 \pm \sqrt{-2304}$.

III. Sit $x^4 + x^2 + 2x + 6 = 0$. Ergo $p = 1$, $q = 2$, $r = 6$; quare sua cubica derivativa $f^6 + 2f^4 - 23f^2 - 4 = 0$, quæ si dividatur per $ff - 4 = 0$, nihil remanet; Ergo $ff = 4$, & $f = \pm 2$; Itaque si accipiatur $f = 2$, erit $g = 2$, $b = 3$; si $f = -2$, erit $g = 3$, $b = 2$; ac proinde radices quæsitæ $1 + \sqrt{-2}$; $1 - \sqrt{-2}$; $-1 + \sqrt{-1}$; $-1 - \sqrt{-1}$; omnes videlicet imaginariæ.

614. *Coroll.* Si cubica derivativa $f^6 + 2f^4 - 23f^2 - 4 = 0$ per $ff - 4 = 0$ dividatur, provenit $f^4 + 6f^2 + 1 = 0$, unde $f^2 = z = -3 \pm \sqrt{8}$. Ergo quum idem in aliis casibus consimilibus accidat, si omnes radices in æquatione biquadratica proposita sint imaginariæ, omnes in cubica derivativa depressa fiunt reales; quare si hujusmodi radices reales forent etiam irrationales, in casum irresolutum relaberemur.

Generalem hic prætermittere non possum quantitatum imaginariarum naturam, quæ generali æquationum omnium proprietati ostendendæ conducit. Sit igitur

615. Propositio VI. *Æquatio qualibet paris dimensionis ex factoribus tantummodo imaginariis coalescens alias habere nequit radices, præter illas, quæ sub forma* $\alpha \pm \beta\sqrt{-1}$, *vel consimili, comprehenduntur.*

Dem.

Dem. Dicta æquatio sit $x^m + px^{m-1} + qx^{m-2} + rx^{m-3} + \&c. = 0$; Quum autem factoribus omnibus imaginariis (*per hyp.*) constet, unum ex ipsis sub denominatione generali $x + \alpha + \beta\sqrt{-1}$ adsumere possum (132. 278. 281.), ergo alterum $x + \alpha - \beta\sqrt{-1}$ æquatio necessario continebit (440.); quare trinomium quoque $x^2 + 2x\beta + \alpha^2 + \beta^2$ includet, adeòque erit per hoc divisibilis. Ponatur peracta divisio. Æquatio proposita evadet $x^{m-2} + p'x^{m-3} + q'x^{m-4} + \&c. = 0$, quæ factores omnes imaginarios & ipsa comprehendet. Hinc si unus ex istis ut ante accipiatur, etiam socius secum adesse debet, ac proinde trinomium rationale & ipsa involvet, per quod erit divisibilis. Igitur post divisionem fiet $x^{m-4} + p''x^{m-5} + q''x^{m-6} + \&c. = 0$; & sic semper, donec exhausto radicum imaginariarum adsumptarum numero, sociæ imaginariæ penitus cum ipsis exhauriantur. Sed nullæ aliæ radices præter imaginarias per formam $B\sqrt[2m]{-1}$, ut jam statuimus (284.), vel per $A + (B\sqrt{-1})^v + C\sqrt[2m]{-1}$ (si quantitati imaginariæ realem adiicere lubeat) exprimi possunt, quæ imaginariæ generatim per formam analogam quan-

quantitati $\alpha \pm \beta\sqrt{-1}$ possunt quoque, ut docuimus, designari. Ergò &c. *Quod erat* &c.

616. *Coroll.* Quicumque igitur sit factorum imaginariorum æquationem rationalem ingredientium numerus, binorum productum erit necessario reale; quum autem omnium hujusmodi factorum numerus esse debeat semper par (ex. gr. $2f$), eorum productum erit $x^{2r} + ax^{2r-1} + bx^{2r-2} +$ &c.

617. Propositio VII. *Æquatio quæcumque in factores reales secundi gradus est sua natura, si non arte huc usque cognita, semper divisibilis.*

Dem. Sit æquatio gradus paris. Ista vel habebit radices omnes reales, vel omnes imaginarias, vel reales cum imaginariis commistas. In primo casu res patet. In secundo quum ob radicum imaginariarum numerum semper parem binorum factorum productum sit per demonstrata in Prop. anteced. trinomium quadraticum reale, pariterque omnium factorum ex hisce trinomiis coalescens sit semper reale, adeoque in trinomia quadratica illud efficientia rursus divisibile, patet propositum. In tertio casu quoniam tam radices reales, quam imaginariæ sunt semper numero pares, ac proinde illarum productum ab istarum producto separabile, redeunt casus primus, & secundus. Ergo &c. *Quod erat primum.*

Sit nunc æquatio gradus imparis. Quum unum saltem habeat factorem realem, per hunc divisa ad æquationem gradus paris reducetur, quæ idcirco in factores reales secundi gradus erit resolubilis. *Quod erat secundum.*

Ele-

Elegantissimi hujus Theorematis usus est insignis in calculo Integrali.

CAPUT X.

DE ÆQUATIONIBUS, QUARUM RADICES FORMAM CARDANICÆ CONSIMILEM EXHIBENT NIMIRUM.

$$\sqrt[n]{\tfrac{1}{2}m+\sqrt{\left(\tfrac{1}{4}m^2+\tfrac{1}{27}p^n\right)}}+$$

$$\sqrt[n]{\tfrac{1}{2}m-\sqrt{\left(\tfrac{1}{4}m^2+\tfrac{1}{27}p^n\right)}}.$$

618. Propositio. *Æquationum radices formam Cardanicæ consimilem recipientium naturam indagare.*

Resolutio. Esto $x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} +$ &c. $= 0$. Quum ex hypothesi esse debeat $x = \sqrt[n]{R} + \sqrt[n]{S}$, si compendii gratia ponatur $\sqrt[n]{R} = r$, $\sqrt[n]{S} = s$, erit $x = r + s$; quare substituendo, proposita æquatio evadet $(r+s)^n + a(r+s)^{n-1} + b(r+s)^{n-2} + c(r+s)^{n-3} +$ &c. $= 0$. Singuli æquationis hujusce termini, adhibita binomii formula Newtoniana, evolvantur, prodibit

I.

I. *Terminus* $r^n + nr^{n-1}s + \frac{n(n-1)}{2}r^{n-2}s^2 +$

$\frac{n(n-1)(n-2)}{2.\ 3}r^{n-3}s^3 + \frac{n(n-1)(n-2)(n-3)}{2.\ 3.\ 4}r^{n-4}s^4$

$+ \&c. + s^n$

II. $ar^{n-1} + a(n-1)r^{n-2}s + \frac{a(n-1)(n-2)}{2}r^{n-3}s^2$

$+ \frac{a(n-1)(n-2)(n-3)}{2.\ 3}r^{n-4}s^3 +$

$\frac{a(n-1)(n-2)(n-3)(n-4)}{2.\ 3.\ 4}r^{n-5}s^4 + \&c. + as^{n-1}.$

III. $br^{n-2} + b(n-2)r^{n-3}s + \frac{b(n-2)(n-3)}{2}r^{n-4}s^2$

$+ \frac{b(n-2)(n-3)(n-4)}{2.\ 3}r^{n-5}s^3 +$

$\frac{b(n-2)(n-3)(n-4)(n-5)}{2.\ 3.\ 4}r^{n-6}s^4 + \&c. + bs^{n-2}.$

IV. $cr^{n-3} + c(n-3)r^{n-4}s + \frac{c(n-3)(n-4)}{2}r^{n-5}s^2$

$+ \frac{c(n-3)(n-4)(n-5)}{2.\ 3}r^{n-6}s^3 +$

$\frac{c(n-3)(n-4)(n-5)(n-6)}{2.\quad 3.\quad 4} r^{n-7} s^4 + \&c. + cs^{n-3}.$

V. $dr^{n-4} + d(n-4) r^{n-5} s + d\frac{(n-4)(n-5)}{2} r^{n-6} s^2$

$+ d\frac{(n-4)(n-5)(n-6)}{2.\quad 3} r^{n-7} s^3 +$

$d\frac{(n-4)(n-5)(n-6)(n-7)}{2.\quad 3.\quad 4} r^{n-8} s^4 + \&c. + ds^{n-4}.$

&c.

Quia vero in poteſtatibus, incipiendo ab ultimo termino & inverſe progrediendo, coëfficientes cum illis congruunt, qui ordine directo procedunt (65. n. V.); ſumptis primo termino r^n, & ultimo s^n; quorum ſumma $r^n + s^n$; ſecundo $nrs^{n-1} = nrs(s^{n-2})$, & penultimo $nsr^{n-1} = nrs(r^{n-2})$, quorum ſumma $nrs(r^{n-2} + s^{n-2})$, & ita porro; termini ſupra inventi in ordinem pro mox dicendis commodiorem ſic diſtribuantur.

$(A) \ldots\ldots r^n + s^n + nrs(r^{n-2} + s^{n-2}) + n\frac{(n-1)}{2} r^2 s^2$

$(r^{n-4} + s^{n-4}) + n\frac{(n-1)(n-2)}{2.\quad 3} r^3 s^3$

Pars alt. R $(r^{n-6}$

$$(r^{n-6}+s^{n-6})+n\frac{(n-1)(n-2)(n-3)}{2\cdot 3\cdot 4}r^4s^4$$

$$(r^{n-8}+s^{n-8})+\&c.$$

$$(B)\ldots\; ar^{n-1}+as^{n-1}+a(n-1)rs(r^{n-3}+s^{n-3})$$

$$+a\frac{(n-1)(n-2)}{2}r^2s^2(r^{n-5}+s^{n-5})+$$

$$a\frac{(n-1)(n-2)(n-3)}{2\cdot 3}r^3s^3(r^{n-7}+s^{n-7})$$

$$+a\frac{(n-1)(n-2)(n-3)(n-4)}{2\cdot 3\cdot 4}r^4s^4$$

$$(r^{n-9}+s^{n-9})+\&c.$$

$$(C)\ldots\; br^{n-2}+bs^{n-2}+b(n-2)rs(r^{n-4}+s^{n-4})$$

$$+b\frac{(n-2)(n-3)}{2}r^2s^2(r^{n-6}+s^{n-6})+$$

$$b\frac{(n-2)(n-3)(n-4)}{2\cdot 3}r^3s^3(r^{n-8}+s^{n-8})$$

$$+b\frac{(n-2)(n-3)(n-4)(n-5)}{2\cdot 3\cdot 4}r^4s^4$$

$$(r^{n-10}+s^{n-10})+\&c.$$

(D)

$(D)\ldots cr^{n-3}+cs^{n-3}+c(n-3)rs(r^{n-5}+s^{n-5})$

$+c\dfrac{(n-3)(n-4)}{2}r^2s^2(r^{n-7}+s^{n-7})+$

$c\dfrac{(n-3)(n-4)(n-5)}{2.\ \ 3}r^3s^3(r^{n-9}+s^{n-9})+$ &c.

$(E)\ldots dr^{n-4}+ds^{n-4}+d(n-4)rs(r^{n-6}+s^{n-6})$

$+d\dfrac{(n-4)(n-5)}{2}r^2s^2(r^{n-8}+s^{n-8})+$

$d\dfrac{(n-4)(n-5)(n-6)}{2.\ \ 3}r^3s^3(r^{n-10}+s^{n-10})+$ &c.

&c.

Jam quoniam $r+s$ eſt æquationis propoſitæ radix [*ex hyp.*], adeoque ejus in propoſitam introductio illam in nihilum redigit; ad hoc comprobandum, oportet, ut omnes radicales elidantur; ergo quum præter r^n, s^n ceteri omnes termini ſint irrationales, rationalibus omiſſis, omnes termini ſimiles, ſeu eodem exponente præditi, ſunt invicem comparandi, ut debita coëfficientium relatio pro radicalium eliſione innoteſcat. Quamobrem (1.°) termini nsr^{n-1} $+nrs^{n-1}$ in primo termino (A) degentes cum ter-

minis $ar^{n-1} + as^{n-1}$ in termino secundo (B) existentibus comparentur, eritque $nr^{n-1}s + nrs^{n-1} = ar^{n-1} + as^{n-1}$. Ut autem duo termini nr^{n-1}, as^{n-1} elidantur, esse debet $a = -ns$; & ut alii duo termini nrs^{n-1}, as^{n-1} evanescant, esse debet $a = -nr$; ex quo fit $r = s$, quod hypothesi adversatur. Ergo quicumque sit valor ipsius a, termini $nr^{n-1}s + nrs^{n-1}$, $ar^{n-1} + as^{n-1}$ destrui nequeunt. At si ponatur $a = 0$, evanescent termini $ar^{n-1} + as^{n-1}$, ceterique reducentur ad formam $nrs(r^{n-2} + s^{n-2})$, qui quum sub adsumpta comparatione non cadant, indicant, propositam æquationem admittere non posse radicem formæ $\sqrt[n]{R} + \sqrt[n]{S}$, nisi secundo termino careat.

(2.°) Quærantur termini, ubi adsunt exponentes $n-2$. In primo termino (A) reperietur $nrs(r^{n-2} + s^{n-2})$, in tertio (C) $br^{n-2} + bs^{n-2}$, nam ob $a = 0$ secundus (B) evanescit. Ergo $-nrs(r^{n-2} + s^{n-2}) =$

br^{n-2}

$br^{n-2} + bs^{n-2}$, ex quo elicitur $b = -nrs$, videlicet fiet hujusmodi terminorum destructio, si fuerit $b = -nrs$, ubi memoriæ inhæreat, esse n quantitatem cum negativo signo conjunctam.

(3.°) Transeamus ad terminos, in quibus occurrunt r^{n-3}, s^{n-3}. Quum autem in primo termino (A) fiet $\frac{n(n-1)(n-2)}{2 \cdot 3} r^3 s^3 (r^{n-6} + s^{n-6}) = \frac{n(n-1)(n-2)}{2 \cdot 3} (s^3 r^{n-3} + r^3 s^{n-3})$; in tertio ($C$) habeatur $b(n-2) rs (r^{n-4} + s^{n-4}) = b(n-2)(sr^{n-3} + rs^{n-3})$; in quarto ($D$) $cr^{n-3} + cs^{n-3}$; facta primæ quantitatis cum duabus reliquis comparatione, obtinetur $-n \frac{(n-1)(n-2)}{2 \cdot 3} r^{n-3} s^3 + r^3 s^{n-3} = b(n-2) r^{n-3} s + rs^{n-3} + cr^{n-3} + cs^{n-3}$, & pro b substituto ejus invento valore $-nrs$, colligitur, debitam elisionem nullo modo perfici posse. Sed si statuatur $c = 0$, duo termini $cr^{n-3} + cs^{n-3}$ evanescent, & reliqui sequentem induent formam $\frac{n(n-1)(n-2)}{2 \cdot 3} r^3 s^3 (r^{n-6} + s^{n-6}) + b(n-2) rs (r^{n-4} + s^{n-4})$,

quæ hùc non pertinet. Ergo æquatio quæsita quarto termino destitui debet.

(4.°) Adeamus modo terminos, qui potestatibus r^{n-4}, s^{n-4} afficiuntur. Et quoniam præter a posuimus quoque $c = 0$, ob quarti termini (D) evanescentiam, erit $-n\frac{(n-1)}{2}r^2s^2\,(r^{n-4}+s^{n-4}) = b(n-2)rs\,(r^{n-4}+s^{n-4})+dr^{n-4}+ds^{n-4}$; quocirca substituendo pro b ejus valorem $-nrs$ jam inventum, prodit $-n\frac{(n-1)}{2}r^2s^2 = -n(n-2)r^2s^2+d$, adeoque $d = n\frac{(n-3)}{2}r^2s^2$.

(5.°) Terminis præterea potestates r^{n-5}, s^{n-5} involventibus primum factis $=0$, deinde inter se collatis, nullo modo eorum elisio haberi potest, adeoque ut æquationis quæsitæ veritas subsistat, ejusdem terminus sextus deficiet.

(6.°) Acceptis itidem terminis, qui continent r^{n-6}, s^{n-6}, habetur æquatio $-n\frac{(n-1)(n-2)}{2.\;\;3}r^3s^3 = b\frac{(n-2)(n-3)}{2}r^2s^2+d(n-4)rs+f$, quare si suffectis pro b, & d eorum valoribus, ponatur $f = -n\frac{(n-4)(n-5)}{2.\;\;3}r^3s^3$, ob $n\frac{(n-2)(n-3)-n(n-3)(n-4)}{2} = n^2-3n$, termini evanescent.

Ita-

Itaque sic prosequendo, omnes termini in locis paribus deficient, & reliquorum coëfficientium valores in locis imparibus invenientur esse

I. n

II. $\frac{n(n-3)}{2}$

III. $\frac{n(n-4)(n-5)}{2 \cdot 3}$

IV. $\frac{n(n-5)(n-6)(n-7)}{2 \cdot 3 \cdot 4}$

V. $\frac{n(n-6)(n-7)(n-8)(n-9)}{2 \cdot 3 \cdot 4 \cdot 5}$

&c.

cujus progressionis lex innotescit. Hinc æquatio generalis æquationes omnes, quarum radices formam Cardanicam recipiunt, repræsentans erit sequens

$$x^n - nrsx^{n-2} + \frac{n(n-3)}{2} r^2s^2x^{n-4} - \frac{n(n-4)(n-5)}{2 \cdot 3} r^3s^3x^{n-6}$$

$$+ \frac{n(n-5)(n-6)(n-7)}{2 \cdot 3 \cdot 4} r^4s^4x^{n-8} \ \&c$$

$$\frac{n(n-6)(n-7)(n-8)(n-9)}{2 \cdot 3 \cdot 4 \cdot 5} r^5s^5x^{n-10} + \&c. \ldots$$

$-r^n - s^n = 0$. *Quod erat* &c.

619. *Corollarium* 1. Si in allatis coëfficientibus n; $\frac{n(n-3)}{2}$; $\frac{n(n-4)(n-5)}{2 \cdot 3}$, &c. subrogetur pro n

numerus exprimens gradum æquationis resolvendæ, quæ terminis in sedibus paribus, ultimo excepto, sit mutilata; atque ista æquatio reperiatur prædita coëfficientibus numericis ab hac substitutione prodeuntibus, dummodo suis in locis adsint species speciebus rs, $r^3 s^3$ &c. proportionales; erit ad æquationes de quibus agitur referenda. Ut autem resolvatur, sat erit tam secundi termini coëfficientem quam ultimum terminum ejusdem comparare cum terminis respondentibus æquationis generalis exhibitæ, ut valores indeterminatarum r, s determinentur, quibus in $x = r + s$ suffectis, radix quæsita obtinebitur, dummodo quoad signa curetur, ut productum rs sit semper rationale.

620. *Coroll.* 2. Inventa una radice, quæ semper haberi poterit, reliquæ obtinentur multiplicando inventam per omnes radices gradus n, quas habet unitas, prout egimus post inventam primam radicem in æquationibus cubicis. Quare quum $z^n - 1 = 0$ unam habeat radicem realem, si fuerit n numerus impar, duasque reales, si n fuerit par, ceteræque sint imaginariæ, tales etiam erunt æquationis resolvendæ radices. Sed in valoribus quantitatum r, & s cum radicibus æquationis $z^n - 1 = 0$ separatim conjungendis stude, ut omnium radicum ex hisce conjunctionibus emergentium productum restituat quantitatem ipsi rs proportionalem, & hanc signo contrario illi, quod in secundo æquationis resolvendæ termino apparet. Quod si æquatio ista resol-

ſolvenda omnes habeat radices reales, reſolutio expoſita caſum irreductibilem involvet.

Exemplum I. Sit $x^5 - 5ax^3 + 5a^2x - b = 0$, quæ eſt æquatio ſpectans ad caſum præſentem. Vi Coroll. 1. erit $5a = 5rs$, ſive $a = rs$, & $b = r^5 + s^5$. Hinc $s = \frac{a}{r}$ & $s^5 = \frac{a^5}{r^5}$, quo valore in æquatione $b = r^5 + s^5$ ſubrogato, provenit $b = \frac{a^5}{r^5} + r^5$; & ordinando per r, habetur $r^{10} - br^5 + a^5 = 0$; adeoque ſtatuendo $r^5 = y$, fit $y^2 - by + a^5 = 0$. Extracta igitur radice quadrata, prodit $y = r^5 = \frac{1}{2}b \pm \sqrt{\left(\frac{1}{4}b^2 - a^5\right)}$, ac proinde $r = \sqrt[5]{\frac{1}{2}b \pm \sqrt{\left(\frac{1}{4}b^2 - a^5\right)}}$. Similiter operando; eruetur $s = \sqrt[5]{\frac{1}{2}b \mp \sqrt{\left(\frac{1}{4}b^2 - a^5\right)}}$, ubi ſigna ſic diſtribuuntur, quod productum rs eſſe debet rationale. Itaque erit $x = r + s = \sqrt[5]{\frac{1}{2}b + \sqrt{\left(\frac{1}{4}b^2 - a^5\right)}} + \sqrt[5]{\frac{1}{2}b - \sqrt{\left(\frac{1}{4}b^2 - a^5\right)}}$. Hæc radix eſt du-

cenda

cenda in omnes radices æquationis $z^5 - 1 = 0$; quæ (dividendo per $z - 1$, ac deinde extrahendo ex quoto radices prout docuimus in Cap. IX.) sunt sequentes

$$1; \frac{-\sqrt{5}-1+\sqrt{-10+2\sqrt{5}}}{4}; \frac{+\sqrt{5}-1+\sqrt{-10-2\sqrt{5}}}{4}$$

$$\frac{-\sqrt{5}-1-\sqrt{-10+2\sqrt{5}}}{4}; \frac{+\sqrt{5}-1-\sqrt{-10-2\sqrt{5}}}{4};$$

ut omnes æquationis datæ radices obtineantur; sed hujusmodi producta ita sunt conjungenda, ut aggregati vel residui partes invicem multiplicatæ præbeant $+a = rs$. Ergo quinque radices quæsitæ erunt

I. $x = r + s$

II. $x = r.\frac{-\sqrt{5}-1+\sqrt{-10+2\sqrt{5}}}{4} + s.\frac{-\sqrt{5}-1-\sqrt{-10+2\sqrt{5}}}{4}$

III. $x = r.\frac{-\sqrt{5}-1-\sqrt{-10+2\sqrt{5}}}{4} + s.\frac{-\sqrt{5}-1+\sqrt{-10+2\sqrt{5}}}{4}$

IV. $x = r.\frac{\sqrt{5}-1+\sqrt{-10-2\sqrt{5}}}{4} + s.\frac{\sqrt{5}-1-\sqrt{-10-2\sqrt{5}}}{4}$

V.

V. $x = r \cdot \frac{\sqrt{5} - 1 - \sqrt{-10 - 2\sqrt{5}}}{4} +$

$s \cdot \frac{\sqrt{5} - 1 + \sqrt{-10 - 2\sqrt{5}}}{4}$

Exemplum II. Sit $x^6 \pm 6ax^4 + 9a^2x^2 - 2c = 0$. Positis $\pm 6a = -6rs$, seu $\mp a = rs$, & $2c = r^6 + s^6$, quum sit $s = \mp \frac{a}{r}$, & $s^6 = \frac{a^6}{r^6}$, erit $2c = r^6 + \frac{a^6}{r^6}$; & ordinando per r, exurget $r^{12} - 2cr^6 + a^6 = 0$. Fac $y = r^6$; prodibit $y^2 - 2cy + a^6 = 0$; unde $y = r^6 = c \pm \sqrt{(c^2 - a^6)}$, & $r = \sqrt[6]{c \pm \sqrt{(c^2 - a^6)}}$. Simili ratione invenietur $s = \sqrt[6]{c \mp \sqrt{(c^2 - a^6)}}$; quamobrem $x = r + s = \sqrt[6]{c \pm \sqrt{(c^2 - a^6)}} + \sqrt[6]{c \mp \sqrt{(c^2 - a^6)}}$. Sed æquationis $x^6 - 1 = 0$ sex radices sunt, $+1$; -1;

$+ \sqrt{-\frac{1}{2} + \frac{1}{2}\sqrt{-3}}$; $- \sqrt{-\frac{1}{2} + \frac{1}{2}\sqrt{-3}}$; $+ \sqrt{-\frac{1}{2} - \frac{1}{2}\sqrt{-3}}$; $- \sqrt{-\frac{1}{2} - \frac{1}{2}\sqrt{-3}}$ (574. n. 4).

Ergo species r, & s ita sunt cum hisce radicibus separatim conjungendæ, ut semper proveniat $rs = +a$, si valeat signum inferius, & $-rs = -a$, si superius. Itaque pro æquatione $x^6 - 6ax^4 + 9a^2x^2 - 2c = 0$ sex nanciscimur radices sequentes

I.

I. $r + s$

II. $-r - s$

III. $r\sqrt{-\frac{1}{2}+\frac{1}{2}\sqrt{-3}} + s\sqrt{-\frac{1}{2}-\frac{1}{2}\sqrt{-3}}$

IV. $-r\sqrt{-\frac{1}{2}+\frac{1}{2}\sqrt{-3}} - s\sqrt{-\frac{1}{2}-\frac{1}{2}\sqrt{-3}}$

V. $-r\sqrt{-\frac{1}{2}-\frac{1}{2}\sqrt{-3}} - s\sqrt{-\frac{1}{2}+\frac{1}{2}\sqrt{-3}}$

VI. $r\sqrt{-\frac{1}{2}-\frac{1}{2}\sqrt{-3}} + s\sqrt{-\frac{1}{2}+\frac{1}{2}\sqrt{-3}}$

quarum singularum partes invicem ductæ præbent $+rs = +a$. Æquationis vero $x^6 + 6ax^4 + 9a^3x^2 - 2c = 0$ radices sunt

I. $r - s$

II. $-r + s$

III. $-r\sqrt{-\frac{1}{2}+\frac{1}{2}\sqrt{-3}} + s\sqrt{-\frac{1}{2}-\frac{1}{2}\sqrt{-3}}$

IV. $r\sqrt{-\frac{1}{2}+\frac{1}{2}\sqrt{-3}} - s\sqrt{-\frac{1}{2}-\frac{1}{2}\sqrt{-3}}$

V. $-r\sqrt{-\frac{1}{2}-\frac{1}{2}\sqrt{-3}} + s\sqrt{-\frac{1}{2}+\frac{1}{2}\sqrt{-3}}$

VI. $r\sqrt{-\frac{1}{2}-\frac{1}{2}\sqrt{-3}} - s\sqrt{-\frac{1}{2}+\frac{1}{2}\sqrt{-3}}$

quod valores r, s inter se ut supra multiplicati dant $-rs = -a$.

IN-

INTRODUCTIO
IN
ALGEBRAM

SECTIO IV.
CAPUT I.
SINUUM DOCTRINA ULTERIUS PROMOTA.

621. Propositio I. *Datis duobus angulis* v, z, *dico, esse*

$$\sin v \cos z = \frac{\sin[v+z]+\sin[v-z]}{2}$$

$$\cos v \cos z = \frac{\sin[v+z]-\sin[v-z]}{2}$$

$$\sin v \sin z = \frac{\cos[v-z]-\cos[v+z]}{2}$$

$$\cos v \cos z = \frac{\cos[v-z]+\cos[v+z]}{2}$$

Dem.

Dem. Etenim $\sin[v \pm z] = \sin v \cos z \pm \cos v \sin z$ (221.). Ergo addendo, ac deinde dividendo per 2, fit $\sin v \cos z = \frac{\sin[v+z] + \sin[v-z]}{2}$ quod æquationem primam oſtendit; reliquæ haud abſimili modo, per ſummam nempe aut ſubtractionem, ac per 2 diviſionem, facile demonſtrantur. *Quod erat* &c.

622. *Coroll.* 1. Si fuerit $v = z = \frac{1}{2} x$; erit $[\cos \frac{1}{2} x]^2 = \frac{1+\cos x}{2}$, & $\cos \frac{1}{2} x = \sqrt{\frac{[1+\cos x]}{2}}$; $[\sin \frac{1}{2} x]^2 = \frac{1-\cos x}{2}$, & $\sin \frac{1}{2} x = \sqrt{\frac{[1-\cos x]}{2}}$. Quod ſi pro $\frac{x}{2}$ ponatur ejus valor duplus x, fiet $\cos x = \sqrt{\frac{[1+\cos 2x]}{2}}$, & $\sin x = \sqrt{\frac{[1-\cos 2x]}{2}}$. Dato nimirum arcu, ſinus & coſinus tam pro ejus dimidio, quam pro ejus duplo haberi facile poſſunt.

623. *Coroll.* 2. Poſito arcu $v + z = a$, & $v - z = b$, erit $v = \frac{a+b}{2}$, & $z = \frac{a-b}{2}$; quare ſubſtituendo fit

$\sin a$

$$\sin a + \sin b = 2 \sin\left[\frac{a+b}{2}\right] \cos\left[\frac{a-b}{2}\right]$$

$$\sin a - \sin b = 2 \cos\left[\frac{a+b}{2}\right] \sin\left[\frac{a-b}{2}\right]$$

$$\cos a + \cos b = 2 \cos\left[\frac{a+b}{2}\right] \cos\left[\frac{a-b}{2}\right]$$

$$\cos b - \cos a = 2 \sin\left[\frac{a+b}{2}\right] \sin\left[\frac{a-b}{2}\right]$$

624. *Coroll.* 3. In duabus primis formulis Coroll. antec. pone $a = 90$; in duabus secundis statue $b = 0$; habebis

$$1 + \sin b = 2 \sin\left[45° + \frac{1}{2} b\right] \cos\left[45° - \frac{1}{2} b\right] =$$

$$2 \sin^2\left[45° + \frac{1}{2} b\right]$$

$$1 - \sin b = 2 \sin\left[45° - \frac{1}{2} b\right] \cos\left[45° + \frac{1}{2} b\right] =$$

$$2 \sin^2\left[45° - \frac{1}{2} b\right] = 2 \cos^2\left[45° + \frac{1}{2} b\right]$$

$$1 + \cos a = 2 \cos^2 \frac{1}{2} a$$

$$1 - \cos a = 2 \sin^2 \frac{1}{2} a,$$

unde per divisionem prodeunt Theoremata

1 +

$$\frac{1+\sin b}{1-\sin b} = \frac{\sin^2[45^\circ+\frac{1}{2}b]}{\sin^2[45^\circ-\frac{1}{2}b]} = \frac{\sin^2[45^\circ+\frac{1}{2}b]}{\cos^2[45^\circ+\frac{1}{2}b]} =$$

$$\mathrm{tang}^2[45^\circ+\frac{1}{2}b]$$

$$\frac{1+\cos a}{1-\cos a} = \frac{\cos^2\frac{1}{2}a}{\sin^2\frac{1}{2}a} = \cot\frac{1}{2}a$$

$$\frac{1+\sin b}{1+\cos a} = \frac{\sin^2[45^\circ+\frac{1}{2}b]}{\cos^2\frac{1}{2}a}$$

$$\frac{1-\sin b}{1-\cos b} = \frac{\sin^2[45^\circ-\frac{1}{2}b]}{\sin^2\frac{1}{2}b}$$

625. *Coroll.* 4. Similiter ex Coroll. 2. per divisionem colliguntur Theoremata

$$\frac{\sin a+\sin b}{\sin a-\sin b} = \mathrm{tang}\,\frac{[a+b]}{2}\cot\frac{[a-b]}{2} =$$

$$\frac{\mathrm{tang}\,\frac{[a+b]}{2}}{\mathrm{tang}\,\frac{[a-b]}{2}}$$

sin a

$$\frac{\sin a + \sin b}{\cos a + \cos b} = \operatorname{tang} \frac{(a+b)}{2}$$

$$\frac{\sin a + \sin b}{\cos b - \cos a} = \cot \frac{(a-b)}{2}$$

$$\frac{\sin a - \sin b}{\cos a + \cos b} = \operatorname{tang} \frac{(a-b)}{2}$$

$$\frac{\sin a - \sin b}{\cos b - \cos a} = \cot \frac{(a+b)}{2}$$

$$\frac{\cos a + \cos b}{\cos b - \cos a} = \cot \frac{(a+b)}{2} \cot \frac{(a-b)}{2};$$

& ex hiſce ſequentia

$$\frac{\sin a + \sin b}{\cos a + \cos b} = \frac{\cos b - \cos a}{\sin a - \sin b}$$

$$\frac{\sin a + \sin b}{\sin a - \sin b} \times \frac{\cos a + \cos b}{\cos b - \cos a} = \left(\cot \frac{(a-b)}{2}\right)^2$$

$$\frac{\sin a + \sin b}{\sin a - \sin b} \times \frac{\cos b - \cos a}{\cos a + \cos b} = \left(\operatorname{tang} \frac{a+b}{2}\right)^2$$

626. Propoſitio II. *Datis duobus angulis* v, z, *dico, productum* $(\cos v \pm \sqrt{-1} \sin v)^n = \cos nv \pm \sqrt{-1} \sin nv$.

Dem. Sumptis enim primum ſignis ſuperioribus, deinde inferioribus, habetur $(\cos v \pm \sqrt{-1} \sin v)$

$(\cos z \pm \sqrt{-1}\,\sin z) = \cos v \cos z - \sin v \sin z \pm (\sin v \cos z + \cos v \sin z)\sqrt{-1}$. Sed $\cos v \cos z - \sin v \sin z = \cos(v+z)$, & $\sin v \cos z + \cos v \sin z = \sin(v+z)$. Ergo $(\cos v \pm \sqrt{-1}\,\sin v)(\cos z \pm \sqrt{-1}\,\sin z) = \cos(v+z) \pm \sqrt{-1}\,\sin(v+z)$. Pone $z = v$. Erit $(\cos v \pm \sqrt{-1}\,\sin v)^2 = \cos 2v \pm \sqrt{-1}\,\sin 2v$. Multiplicetur rursus eadem formula per $\cos z \pm \sqrt{-1}\,\sin z$; factum erit $\cos(2v+z) \pm \sqrt{-1}\,\sin(2v+z)$; quare si statuatur $z = v$, prodibit

$(\cos v + \sqrt{-1}\,\sin v)^3 = \cos 3v \pm \sqrt{-1}\,\sin 3v$.

Quod si hæc expressio denuo ducatur in $\cos z + \sqrt{-1}$ $\sin z$, fiatque postea $z = v$, proveniet factum

$(\cos v + \sqrt{-1}\,\sin v)^4 = \cos 4v \pm \sqrt{-1}\,\sin 4v$.

Ergo generatim $(\cos v + \sqrt{-1}\,\sin v)^n = \cos nv \pm \sqrt{-1}\,\sin nv$. *Quod erat* &c.

627. *Coroll.* 1. Si agatur de sinu vel cosinu residui $v - z$, quum sit $-\sin z = \sin - z$, & $\cos - z = \cos z$, sat erit ipsius z signum mutare. Sic $(\cos v - \sqrt{-1}\,\sin v)(\cos z + \sqrt{-1}\,\sin z) = \cos(v-z) + \sin(v-z)$.

628. *Coroll.* 2. Igitur $\sin nv\sqrt{-1} = (\cos v + \sin v\sqrt{-1})^n - \cos nv$, & $\sin nv\sqrt{-1} = \cos nv - (\cos v - \sin v\sqrt{-1})^n$, quibus valoribus additis, factaque per $2\sqrt{-1}$ divisione, prodit $\sin nv =$

$$\frac{(\cos v + \sin v\sqrt{-1})^n - (\cos v - \sin v\sqrt{-1})^n}{2\sqrt{-1}}.$$

Si-

Simili operatione quoad cosinus nv instituta, reperitur

$$\cos nv = \frac{(\cos v + \sin v\sqrt{-1})^n + (\cos v - \sin v\sqrt{-1})^n}{2}.$$

629. *Coroll.* 3. Quocirca si ex æquatione $(\cos v \pm \sqrt{-1}\sin v)^n = \cos nv \pm \sqrt{-1}\sin nv$ extrahatur radix n, fient duæ æquationes

$$\cos v + \sqrt{-1}\sin v = (\cos nv + \sqrt{-1}\sin nv)^{\frac{1}{n}}$$

$$\cos v - \sqrt{-1}\sin v = (\cos nv - \sqrt{-1}\sin nv)^{\frac{1}{n}},$$

& primum addendo habetur

$$\cos v = \frac{(\cos nv + \sqrt{-1}\sin nv)^{\frac{1}{n}} + (\cos nv - \sqrt{-1}\sin nv)^{\frac{1}{n}}}{2},$$

deinde subtrahendo.

$$\sin v = \frac{(\cos nv + \sqrt{-1}\sin nv)^{\frac{1}{n}} - (\cos nv - \sqrt{-1}\sin nv)^{\frac{1}{n}}}{2\sqrt{-1}}.$$

630. *Coroll.* 4. Itaque $\tang v = \frac{\sin v}{\cos v} =$

$$\frac{(\cos nv + \sqrt{-1}\sin nv)^{\frac{1}{n}} - (\cos nv - \sqrt{-1}\sin nv)^{\frac{1}{n}}}{2\sqrt{-1}\left((\cos nv + \sqrt{-1}\sin nv)^{\frac{1}{n}} + (\cos nv - \sqrt{-1}\sin nv)^{\frac{1}{n}}\right)};$$

adeoque etiam per calculum ostenditur, eundem sinum, vel cosinum, aliasque circuli functiones infinitos valores habere posse, prout jam indicavimus (215.).

631. *Coroll.* 5. Si ponamus $\cos v + \sqrt{-1}\,\sin v = p$, & $\cos v - \sqrt{-1}\,\sin v = q$, ob $p^n = \cos nv + \sqrt{-1}\,\sin nv$, & $q^n = \cos nv - \sqrt{-1}\,\sin nv$, erit $p^n + q^n = 2\cos nv$, & $p^n - q^n = 2\sqrt{-1}\,\sin nv$, nec non $pq = 1$, cujus transformationis usus infra innotescet.

632. *Coroll.* 6. Duobus ultimis Coroll. secundi binomiis ad potestatem n per formulam Newtonianam elevatis, factisque debitis divisionibus, habetur

$$\text{I. } \sin nv = \frac{n}{1}(\cos v)^{n-1}\sin v - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}(\cos v)^{n-3}(\sin v)^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot 2\cdot 3\cdot 4\cdot 5}(\cos v)^{n-5}(\sin v)^5 - \&c.$$

$$\text{II. } \cos nv = (\cos v)^n - \frac{n(n-1)}{1\cdot 2}(\cos v)^{n-2}(\sin v)^2 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}(\cos v)^{n-4}(\sin v)^4 - \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}(\cos v)^{n-6}(\sin v)^6 + \&c.$$

633.

633. *Coroll.* 7. Quia $\frac{\sin nv}{\cos nv} = \tan g\, nv =$

$$\frac{\cos^n v\left(n\cos^{-1} v \sin v - \frac{n(n-1)(n-2)}{2.\ 3}\cos^{-3} v \sin^3 v + \frac{n(n-1)(n-2)(n-3)(n-4)}{2.\ 3.\ 4.\ 5}\cos^{-5} v \sin^5 v\ \&c.\right)}{\cos^n v\left(1 - \frac{n(n-1)}{2}\cos^{-2} v \sin^2 v + \frac{n(n-1)(n-2)(n-3)}{2.\ 3.\ 4}\cos^{-4} v \sin^4 v\ \&c.\right)}$$

erit $\tan g\, nv =$

$$\frac{n \tan g\, v - \frac{n[n-1][n-2]}{2.\ 3}\tan g^3 v + \frac{n[n-1][n-2][n-3][n-4]}{2.\ 3.\ 4.\ 5}\tan g^5 v - \&c.}{1 - \frac{n[n-1]}{2}\tan g^2 v + \frac{n[n-1][n-2][n-3]}{2.\ 3.\ 4}\tan g^4 v - \&c.}$$

nam $(\sin v)^n$ exprimi etiam poteſt per $\sin^n v$, vel per $\sin v^n$ & $(\cos v)^n$ per $\cos^n v$, vel per $\cos v^n$.

634. Propoſitio III. *Dato angulo v, tam ejus ſinum, quam coſinum per ſeries exprimere.*

Reſolutio. Quoniam inſpectis formulis in Coroll. ſexto exhibitis, prima intuitu ſinuum conſtat terminis dignitatum imparium binomii $(\cos v + \sin v)^n$, ſecunda terminis dignitatum parium ejuſdem binomii, ac prima habet $\sin v$ in primo termino, ſecunda unitatem; ponamus

$$\sin v = v + av^3 + bv^5 + cv^7 + \&c.$$

$$\cos v = 1 + Av^2 + Bv^4 + Cv^6 + \&c.$$ Erit

$$1 - \sin^2 v = 1 - v^2 - 2av^4 - a^2v^6 - \&c. \\ - 2bv^6$$

$$\cos^2 v = 1 + 2Av^2 + A^2v^4 + 2Cv^6 + \&c. \\ + 2Bv^4 + 2ABv^6$$

$$\cos 2v = 1 + 4Av^2 + 16Bv^4 + 64Cv^6 + \&c.$$

Jam ob $\cos^2 v = 1 - \sin^2 v$, & $2 \sin v \cos v = \sin 2v$, ſi fiat $\cos^2 v = P$; $2 \sin v \cos v = Q$; $1 - \sin^2 v = R$; $\sin 2v = S$; erit $P + Q = R + S$, ſive

$$1 + 2v + 2Av^2 + 2Av^3 + A^2v^4 + 2Aav^5 + 2Cv^6 + \&c. \\ + 2av^3 + 2Bv^4 + 2Bv^5 + 2ABv^6 \\ + 2bv^5$$

$$= 1 + 2v - v^2 + 8av^3 - 2av^4 + 32bv^5 - a^2v^6 - \&c. \\ - 2bv^6$$

Itaque ſi termini homologi comparentur, invenietur

$$A = -\frac{1}{2};\ a = -\frac{1}{2 \cdot 3};\ B = \frac{1}{2 \cdot 3 \cdot 4};\ b =$$

$-\frac{1}{2\cdot3\cdot4\cdot5}$; $C = -\frac{1}{2\cdot3\cdot4\cdot5\cdot6}$; &c. Ergo

$$\sin v = v - \frac{v^3}{1\cdot2\cdot3} + \frac{v^5}{1\cdot2\cdot3\cdot4\cdot5} - \frac{v^7}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7} + \&c.$$

$$\cos v = 1 - \frac{v^2}{1\cdot2} + \frac{v^4}{1\cdot2\cdot3\cdot4} - \frac{v^6}{1\cdot2\cdot3\cdot4\cdot5\cdot6}$$

$+$ &c. *Quod erat* &c.

635. *Coroll.* 1. Ergo ex jam demonstratis (270. 271.) si ponatur *e* numerus, cujus logarithmus sit unitas, erit

$$\frac{e^{v\sqrt{-1}} - e^{-v\sqrt{-1}}}{2\sqrt{-1}} = \sin v$$

$$\frac{e^{v\sqrt{-1}} + e^{-v\sqrt{-1}}}{2} = \cos v;$$

ac proinde $\frac{e^{2v\sqrt{-1}} - e^{-2v\sqrt{-1}}}{4\sqrt{-1}} = \sin v \cos v = \frac{\sin 2v}{2}$ (621.)

En quantitates exponentiales imaginarias ad sinus, & cosinus arcuum realium reductas.

636. *Coroll.* 2. Quum autem sit $\mathit{sin}\, v\sqrt{-1} = \frac{e^{+v\sqrt{-1}} - e^{-v\sqrt{-1}}}{2}$, & $\mathit{cos}\, v = \frac{e^{v\sqrt{-1}} + e^{-v\sqrt{-1}}}{2}$, erit $\mathit{cos}\, v + \mathit{sin}\, v\sqrt{-1} = e^{v\sqrt{-1}}$. Pari ratione oftenditur $e^{-v\sqrt{-1}} = \mathit{cos}\, v - \mathit{sin}\, v\sqrt{-1}$.

Idem demonftrari quoque poteft per æquationem $e^{v} = 1 + \frac{v}{1} + \frac{v^2}{1 \cdot 2} + \frac{v^3}{1 \cdot 2 \cdot 3} + \&c.$ jam exhibitam (268.) fubftituendo $\pm v\sqrt{-1}$ in locum v, & feparando quantitates attinentes ad $\mathit{cos}\, v$, & ad $\mathit{sin}\, v\sqrt{-1}$.

637. *Coroll.* 3. Ob $\frac{\mathit{sin}\, v}{\mathit{cos}\, v} = \mathit{tang}\, v$, erit $\mathit{tang}\, v$

$$= \frac{e^{v\sqrt{-1}} - e^{-v\sqrt{-1}}}{\left(e^{v\sqrt{-1}} + e^{-v\sqrt{-1}}\right)\sqrt{-1}}.$$

Suffice $x\sqrt{-a}$ in locum v, ac fiet $\mathit{sin}\, x\sqrt{-a} = \frac{e^{-x\sqrt{a}} - e^{x\sqrt{a}}}{2\sqrt{-1}}$; $\mathit{cos}\, x\sqrt{-a} = \frac{e^{-x\sqrt{a}} + e^{x\sqrt{a}}}{2}$; quare $\mathit{tang}\, x\sqrt{-a}$

$$= \frac{e^{-x\sqrt{a}} - e^{x\sqrt{a}}}{\left(e^{-x\sqrt{a}} + e^{x\sqrt{a}}\right)\sqrt{-1}};$$

& $\frac{x\sqrt{-a}}{\mathit{tang}\, x\sqrt{a}} =$

(e

$$\frac{\left(e^{-x\sqrt{a}} + e^{x\sqrt{a}}\right)x\sqrt{a}}{e^{+x\sqrt{a}} - e^{-x\sqrt{a}}}\text{ ; adeoque } cos\ x\sqrt{-a} =$$

$$\frac{\left(e^{-x\sqrt{a}} + e^{x\sqrt{a}}\right)\sqrt{-1}}{e^{-x\sqrt{a}} - e^{x\sqrt{a}}}.$$

638. *Corollarium* 5. Similiter

$$tang\ v = \frac{v - \frac{v^3}{1.2.3} + \frac{v^5}{1.2.3.4.5} - \frac{v^7}{1.2.3....7} + \&c.}{1 - \frac{v^2}{1.2} + \frac{v^4}{1.2.3.4} - \frac{v^6}{1.2.3....6} + \&c.}$$

$$cos\ v = \frac{1 - \frac{v^2}{1.2} + \frac{v^4}{1.2.3.4} - \frac{v^6}{1.2.3....6} + \&c.}{v - \frac{v^3}{1.2.3} + \frac{v^5}{1.2.3.4.5} - \frac{v^7}{1.2.3....7} + \&c.}$$

639. *Scholion.* Ex formulis præmissis (635. 636.) nonnulla haud omittenda colliguntur, nimirum

(1.°) Æquationis $x^m \pm a^m$ factores sic eliciuntur.

In æquatione $cos\ v = \frac{e^{v\sqrt{-1}} + e^{-v\sqrt{-1}}}{2}$ pono v

=

$= \frac{(2k+1)}{m}\pi$, ubi k numerum quemvis integrum, & π semicircumferentiam repræsentant; fiet

$$\cos\frac{(2k+1)}{m}\pi = \frac{e^{\frac{(2k+1)}{m}\pi\sqrt{-1}} + e^{\frac{-(2k+1)}{m}\pi\sqrt{-1}}}{2}.$$

Sed quia $e^{\pi\sqrt{-1}} = \cos\pi + \sqrt{-1}\sin\pi = -1$; ob $\sin\pi = 0$; erit etiam $e^{(2k+1)\pi\sqrt{-1}} = -1$; elevatur enim $e^{\pi\sqrt{-1}}$ ad potestatem imparem; adeoque $e^{\frac{(2k+1)}{m}\pi\sqrt{-1}} = \sqrt[m]{-1}$. Hinc $\cos\frac{(2k+1)}{m}\pi = \frac{\sqrt[m]{-1} + \frac{1}{\sqrt[m]{-1}}}{2}$. Interea si fiat $\sqrt[m]{-1} = x$, erit $x^m + 1 = 0$, & $\cos\frac{(2k+1)}{m}\pi = \frac{x^2+1}{2x}$. Quum autem x exprimat radices æquationis $x^m + 1 = 0$, substitutis hujusmodi radicibus in formula $\frac{x^2+1}{2x}$, exur-

exurgent valores ipsius $cos\,\frac{(2k+1)}{m}\pi$. Sic si fuerit $m=3$, proveniet $cos\,(2k+1)\,60^\circ$, & æquationis $x^3+1=0$ radices erunt $x=-1$, $x=\frac{1}{2}+\frac{1}{2}\sqrt{-3}$, $x=\frac{1}{2}-\frac{1}{2}\sqrt{-3}$, quibus in $\frac{x^2+1}{2x}$ subsectis, habentur tres valores ejusdem $cos\,(2k+1)\,60^\circ$, nempe -1, $+\frac{1}{2}$, $+\frac{1}{2}$, nec plures esse possunt, quicumque enim statuatur numerus integer k, iidem cosinus semper redibunt. Hisce manentibus, quoniam $cos\,\frac{(2k+1)}{m}\pi=\frac{x^2+1}{2x}$ reducitur ad $x^2-2x\,cos\,\frac{(2k+1)}{m}\pi+1=0$, liquet, hunc esse factorem quadraticum æquationis $x^m+1=0$, pariterque $x^2-2ax\,cos\,\frac{(2k+1)}{m}\pi+a^2=0$ esse factorem quadraticum generalem æquationis $x^m+a^m=0$. Numero igitur m determinato, sumptisque cum zero pro k numeris omnibus integris, donec k non superet m (secus iidem sinus rursus occurrerent), facto-

ctores omnes, ac proinde omnes æquationis $x^m + a^m = 0$ radices emergent. Sit ex. gr. $x^7 + a^7 = 0$; hujus æquationis factor generalis erit $x^2 - 2ax \cos\frac{(2k+1)}{7}\pi + a^2 = 0$. Itaque ponendo successive $k = 0$, $k = 1$, $k = 2$, $k = 3$, prodeunt factores $x^2 - 2ax \cos\frac{1}{7}\pi + a^2 = 0$; $x^2 - 2ax \cos\frac{3}{7}\pi + a^2 = 0$; $x^2 - 2ax \cos\frac{5}{7}\pi + a^2 = 0$; $x^2 + 2ax + a^2 = 0$, qui ad simpliciores terminos reductus evadit $x + a = 0$. Cognitis igitur per circulum $\cos\frac{1}{7}\pi$, $\cos\frac{3}{7}\pi$, $\cos\frac{5}{7}\pi$, æquatio resolvitur.

(2.°) Haud absimili ratiocinio reperiuntur factores æquationis $x^m - a^m = 0$. Nam in æquatione $2\cos v = e^{v\sqrt{-1}} + e^{-v\sqrt{-1}}$, ponendo $v = \frac{2k\pi}{m}$, obtineo $2\cos\frac{2k\pi}{m} = e^{\frac{2k}{m}\pi\sqrt{-1}} + e^{\frac{-2k}{m}\pi\sqrt{-1}}$, sive ob

ob $e^{\pi\sqrt{-1}} = -1$, habeo $e^{2k\pi\sqrt{-1}} = +1$, & $e^{\frac{2k}{m}\pi\sqrt{-1}} = \sqrt[m]{1}$. Esto $\sqrt[m]{1} = x$; fiet $2\cos\frac{2k\pi}{m} = \frac{x^2+1}{x}$, quare $x^2 - 2x\cos\frac{2k}{m}\pi + 1 = 0$ erit factor generalis æquationis $x^m - 1 = 0$, & $x^2 - 2ax\cos\frac{2k}{m}\pi + a^2 = 0$ factor generalis æquationis $x^m - a^m = 0$. Sit ex. gr. $x^7 - a^7 = 0$. Subrogatis successive pro k numeris integris $0, 1, 2, 3$, septem obtinentur factores per æquationes $x^2 - 2ax + a^2 = 0$, sive $x - a = 0$; $x^2 - 2ax\cos\frac{2}{7}\pi + a^2 = 0$; $x^2 - 2ax\cos\frac{4}{7}\pi + a^2 = 0$; $x^2 - 2ax\cos\frac{6}{7}\pi + a^2 = 0$.

(3.°) Quantitatum $e^{v\sqrt{-1}}$, $e^{-v\sqrt{-1}}$ sumptis logarithmis, habetur $\pm v\sqrt{-1} = l(\cos v \pm \sqrt{-1}\sin v)$; quapropter si fiat $v = \pm(2k+1)\pi$, ubi sit k, ut supra, numerus quilibet integer, & π semicircumferentiam designet; ob numerum $2k+1$ semper imparem, erit $\sin v = 0$, $\cos v = -1$, adeoque $\pm(2k+1)\pi\sqrt{-1} = l-1$, iidemque ob $l-a = la + l-1$, si ponatur $la = A$, erit $l-a = A \pm (2k+1)\pi\sqrt{-1}$; quod indicat (suffecto successive pro k numero integro quocumque, non omisso zero), lo-

logarithmum quantitatis -1, vel $-a$, infinitos habere valores, omnes imaginarios, ut idem sinus, vel cosinus infinitos arcus, diversos sibi respondentes agnoscit.

(4.°) Quod si in formula $l(\cos v \pm \sqrt{-1}\sin v) = \pm v\sqrt{-1}$ ponatur $v = 2k\pi$; propter numerum $2k$ semper parem, erit $\sin v = 0$, $\cos v = 1$, ex quo fit $l1 = \pm 2k\pi\sqrt{-1}$; vel ob $la = la + l1$ (ponendo ut ante $la = A$) erit $la = A \pm 2k\pi\sqrt{-1}$. Quamobrem logarithmus quantitatis $+1$, vel $+a$ infinitos habet valores omnes imaginarios præter unum realem (nempe o si fiat $a = 1$) qui nascitur, si statuatur $k = 0$.

(5.°) Ex hisce tandem concludimus, quamvis tam $l-1$, quam $l+1$ infinitis gaudeant logarithmis, unitatis attamen negativæ logarithmos semper diversos emergere a logarithmis unitatis positivæ; substitutis siquidem pro k enunciatis valoribus, formula $\pm 2k\pi\sqrt{-1}$, quicumque in ea terminus accipiatur, nunquam cum aliquo ex formulæ $\pm(2k+1)\pi\sqrt{-1}$ terminis potest concurrere.

(6.°) Obiiciet fortasse aliquis, tam $2l-a$, quam $2l+a$ adæquare la^2, quod a^2 tam exurgit ex $-a \times -a$, quam ex $+a \times +a$, adeoque esse $l-a = l+a$. Sed advertendum, radices $-a$, & $+a$, ex quarum in se ipsas multiplicatione quadratum a^2 consurgit, non esse (ut sub hujus Operis initium demonstravimus) invicem æquales; pari ratione quamvis sit $l(-a)^2 = l(+a)^2$, non ideo consequitur, esse $l-a = l+a$, si enim non est $-a = +a$, neque erit $l-a = l+a$, quod etiam ea generaliter confirmant, quæ in numero

mero antecedente commemoravimus. Idem valet ratiocinium, si quis instaret, esse $la^2 = l-a + l-a = l+a + l+a$, & idcirco $l-a = l+a$; nam negari nequit, esse $la^2 = l-a + l-a$, & $la^2 = l+a + l+a$, at hinc rursus non sequitur, esse $l-a + l-a = l+a + l+a$, sive $l-a = l+a$, adeoque $-a = +a$, secus id quod debetur adæquaret omnino id, quod possidetur, quod est falsum. In hoc igitur casu deficit axioma, *quæ sunt eadem uni tertio, sunt idem inter se.*

640. Propositio IV. *Logarithmum imaginarium* $\frac{1}{2\sqrt{-1}}\, l\, \frac{\cos v + \sin v\sqrt{-1}}{\cos v - \sin v\sqrt{-1}}$ *ad arcum realem revocare.*

Resolutio. Quoniam ostendimus (637.), esse $e^{v\sqrt{-1}} = \cos v + \sin v\sqrt{-1}$; sumptis logarithmis hyperbolicis (in quibus $le = 1$), stabit $v\sqrt{-1} = l(\cos v + \sin v\sqrt{-1})$; Pari modo habetur $-v\sqrt{-1} = l(\cos v - \sin v\sqrt{-1})$; Ergo subtrahendo fit $2v\sqrt{-1} = l(\cos v + \sin v\sqrt{-1}) - l(\cos v - \sin v\sqrt{-1})$, ac tandem $v = \frac{1}{2\sqrt{-1}}\, l\, \frac{\cos v + \sin v\sqrt{-1}}{\cos v - \sin v\sqrt{-1}}$. *Quod erat &c.*

641. *Coroll.* 1. Si $\frac{1}{2\sqrt{-1}}\, l\, \frac{\cos v + \sqrt{-1}\sin v}{\cos v - \sqrt{-1}\sin v}$ dividatur supra, & infra per $\cos v$; ob $\frac{\sin v}{\cos v} = \tang v$,

fiet

fiet $v = \frac{1}{2\sqrt{-1}} \, l \frac{1+\sqrt{-1}\, tang\, v}{1-\sqrt{-1}\, tang\, v}$. At ex dictis

(258.) habemus $l \frac{1+z}{1-z} = 2z + \frac{2z^3}{3} + \frac{2z^5}{5} +$

$\frac{2z^7}{7}$ + &c. Ergo si statuatur $z = \sqrt{-1}\, tang\, v$,

emerget $v = \frac{1}{2\sqrt{-1}} \, l \frac{1+z}{1-z} = \frac{tang}{1} v -$

$\frac{(tang\, v)^3}{3} + \frac{(tang\, v)^5}{5} - \frac{(tang\, v)^7}{7}$ + &c. Igitur si ponatur $v =$ arcui $45°$, sive $v = \frac{\pi}{4}$, statuendo $\pi =$ semicircumferentiæ; quoniam tunc tangens fit $=$ radio $= 1$, stabit $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} +$ &c. quæ est series a celeberrimis Viris *Gregorio*, & *Leibnitzio* excogitata ad circumferentiæ valorem exprimendum.

642. *Coroll.* 2. Stante $v =$ arc. $45°$, adeoque $tang\, v = 1$; quum tota circumferentia sit $8v$, erit

$$8v = \frac{4}{\sqrt{-1}} \, l \frac{1+\sqrt{-1}}{1-\sqrt{-1}} = 2 \cdot 2 \frac{l\sqrt{-1}}{\sqrt{-1}} = 2 \frac{l(\sqrt{-1})^2}{\sqrt{-1}}$$

$=$

$= \frac{2l-1}{\sqrt{-1}}$; quare circumferentia integra erit ad diametrum in ratione $l-1:\sqrt{-1}$, prout jam invenit summus Geometra *Joh. Bernoullius.*

643. *Coroll.* 3. Hinc $l-1=4\sqrt{-1}(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\&c.)$; quod rursus demonstrat, logarithmos quantitatum negativarum esse imaginarios. Sed brevius $l(\sqrt{-1})^2 = l-1 = 2l\sqrt{-1}$.

644. Propositio V. *Expressionem* cos nv *in sinuum, & cosinuum* z *expressionem convertere.*

Resolutio. Ponatur $v = 90^\circ - z$. Quoniam $\cos nv = \cos n(90^\circ - z) = \cos n\,90^\circ \cos nz \pm \sin n\,90^\circ \sin nz$; itatuendo pro n successive numeros 1, 2, 3, 4, &c. nanciscimur $\cos v = \sin z$; $\cos 2v = -\cos 2z$; $\cos 3v = -\sin 3z$; $\cos 4v = \cos 4z$; $\cos 5v = \sin 5z$; $\cos 6v = -\cos 6z$; $\cos 7v = -\sin 7z$; &c. *Quod erat* &c.

645. Propositio VI. *Cosinus cujuslibet potestates per cosinus simplices exprimere.*

Resolutio. Potestate n cosinus v, per $\cos v^n$ designata, positisque $\cos v + \sqrt{-1}\cdot\sin v = p$, & $\cos v - \sqrt{-1}\cdot\sin v = q$ (631.), erit $2\cos v = p+q$, & $2^n \cos v^n = (p+q)^n$. Evolvatur primum $(p+q)^n$, deinde $(q+p)^n$ per formulam Newtonianam, tum facta formularum additione, habebitur

646.

$$2^{n}\cos v^{n} = p^{n} + \frac{n}{1}p^{n-1}q + \frac{n(n-1)}{1.\ 2}p^{n-2}q^{2} + \frac{n(n-1)(n-2)}{1.\ 2.\ 3}p^{n-3}q^{3} + \&c.$$

$$2\cos v^{n} = q^{n} + \frac{n}{1}q^{n-1}p + \frac{n(n-1)}{1.\ 2}q^{n-2}p^{2} + \frac{n(n-1)(n-2)}{1.\ 2.\ 3}q^{n-3}p^{3} + \&c.$$

$$2^{n+1}\cos v^{n} = p^{n} + q^{n} + \frac{n}{1}\left(p^{n-2} + q^{n-2}\right)pq + \frac{n(n-1)}{1.\ 2}\left(p^{n-4} + q^{n-4}\right)p^{2}q^{2} + \&c.$$

ubi facta per 2 divisione, ob $pq = 1$, prodibit

$$2^{n}\cos v^{n} = \frac{1}{2}\left(p^{n} + q^{n}\right) + \frac{n}{1}\cdot\frac{1}{2}\left(p^{n-2} + q^{n-2}\right) + \frac{n(n-1)}{1.\ 2}\cdot\frac{1}{2}\left(p^{n-4} + q^{n-4}\right) + \&c.$$

substitue $2\cos nv$ pro æquali $p^{n} + q^{n}$, ac tandem emerget

(A).... $2^{n}\cos v^{n} = \cos nv + \frac{n}{1}\cos(n-2)v + \frac{n(n-1)}{1.\ 2}\cos(n-4)v + \frac{n(n-1)(n-2)}{1.\ 2.\ 3}\cos(n-6)v$ $+ \&c.$ *Quod erat* &c.

646. *Coroll.* 1. Sit jam $n = 1$; quoniam $\cos(n-n)v = \cos(m-n)v$ (214. n. 5.) & $\cos 0 v = 1$, erit $2 \cos v = \cos v + \cos v = 2 \cos v$, adeoque $\cos v = \cos v$. Sit $n = 2$; habetur $2^2 \cos v^2 = \cos 2v + 2 \cos 0 v + \cos 2v = 2 \cos 2v + 2$. Sit $n = 3$; fiet $2^3 \cos v^3 = \cos 3v + 3 \cos v + 3 \cos v + \cos 3v = 2 \cos 3v + 6 \cos v$. Sit $n = 4$; erit $2^4 \cos v^4 = \cos 4v + 4 \cos 2v + 6 \cos 0 v + 4 \cos 2v + \cos 4v = 2 \cos 4v + 8 \cos 2v + 6$; &c.

Ratio est, quia quadratum tres habet terminos, cubus quatuor, biquadratum quinque, &c. quare tot termini in expressione generali (A) sunt in casibus particularibus accipiendi, quot indicat $n + 1$.

647. *Coroll.* 2. Posito igitur n numero integro positivo, factaque divisione per 2, habetur

$$1. \cos v = \cos v$$

$$2. \cos v^2 = \cos 2v + \frac{1}{2} . 2$$

$$4. \cos v^3 = \cos 3v + 3 \cos v$$

$$8. \cos v^4 = \cos 4v + 4 \cos 2v + \frac{1}{2} . 6$$

$$16. \cos v^5 = \cos 5v + 5 \cos 3v + 10 \cos v$$

$$32. \cos v^6 = \cos 6v + 6 \cos 4v + 15 \cos 2v + \frac{1}{2} . 20$$

$$64. \cos v^7 = \cos 7v + 7 \cos 5v + 21 \cos 3v + 35 \cos v$$

$$182. \cos v^8 = \cos 8v + 8 \cos 6v + 28 \cos 4v + 56 \cos 2v + \frac{1}{2} 70 \text{ \&c.}$$

648. *Coroll.* 3. Si vero n sit numerus negativus, series (A) evadit infinita sic

$$\frac{1}{2\ cos\ v} = cos\ v - cos\ 3v + cos\ 5v - cos\ 7v + cos\ 9v - \&c.$$

$$\frac{1}{4\ cos\ v^2} = cos\ 2v - 2\ cos\ 4v + 3\ cos\ 6v - 4\ cos\ 8v - 5\ cos\ 10v - 6\ cos\ 12v + \&c.$$

$$\frac{1}{8\ cos\ v^3} = cos\ 3v - 3\ cos\ 5v + 6\ cos\ 7v - 10\ cos\ 9v + 15\ cos\ 11v - 21\ cos\ 13v + \&c.$$

$$\frac{1}{16\ cos\ v^4} = cos\ 4v - 4\ cos\ 6v - 10\ cos\ 8v - 20\ cos\ 10v + 35\ cos\ 12v - 56\ cos\ 14v + \&c.$$

&c.

649. *Coroll.* 4. Quod si n fuerit numerus fractus, proveniet

$$\sqrt{2\ cos\ v} = cos\ \frac{1}{2}v + \frac{1}{2}\ cos\ \frac{3}{2}v - \frac{1 \cdot 1}{2 \cdot 4}\ cos\ \frac{7}{2}v + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}\ cos\ \frac{11}{2}v - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}\ cos\ \frac{15}{2}v + \&c.$$

$$\frac{1}{\sqrt{2\ cos\ v}} = cos\ \frac{1}{2}v - \frac{1}{2}\ cos\ \frac{5}{2}v + \frac{1 \cdot 3}{2 \cdot 4}\ cos\ \frac{9}{2}v - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\ cos\ \frac{13}{2}v + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}\ cos\ \frac{17}{2}v - \&c.$$

ubi coëfficientes iidem sunt cum illis, qui in extractione radicis per formulam Newtonianam obtinentur.

650.

650. *Scholion.* Præstat interdum formulas Corollarii secundi commodius ordine inverso disponere, ipsas in duas classes, prout n sit par vel impar, distribuendo. Sic si n sit numerus par, fient

$$2\cos v^2 = 1 + \cos 2v$$

$$8\cos v^4 = 3 + 4\cos 2v + \cos 4v$$

$$32\cos v^6 = 10 + 15\cos 2v + 6\cos 4v + \cos 6v$$

$$128\cos v^8 = 35 + 56\cos 2v + 28\cos 4v + 8\cos 6v + \cos 8v$$

$$512\cos v^{10} = 126 + 210\cos 2v + 120\cos 4v + 45\cos 6v + 10\cos 8v + \cos 10v$$

$$2048\cos v^{12} = 462 + 792\cos 2v + 495\cos 4v + 220\cos 6v + 66\cos 8v + 12\cos 10v + \cos 12v$$

&c.

Generatim si fuerit $n = 2\nu$, habebitur

$$2^{2\nu-1}\cos v^{2\nu} = \frac{1}{2}\cdot\frac{2\nu(2\nu-1)(2\nu-2)\ldots\ldots(\nu+1)}{1.\ \ 2.\ \ 3\ldots\ldots\ldots\ldots\nu}$$

$$+\frac{2\nu(2\nu-1)(2\nu-2)\ldots\ldots(\nu+2)}{1.\ \ 2.\ \ 3\ \ldots\ldots\ (\nu-1)}\cos 2v +$$

$$+\frac{2\nu(2\nu-1)(2\nu-2)\ldots\ldots(\nu+3)}{1.\ \ 2.\ \ 3\ldots\ldots\ldots(\nu-2)}\cos 4v +$$

$$\frac{2\nu(2\nu-1)(2\nu-2)\ldots\ldots(\nu+4)}{1.\ \ 2.\ \ 3\ldots\ldots\ldots(\nu-3)}\cos 6v +$$

$$+\frac{2\nu(2\nu-1)(2\nu-2)\ldots\ldots(\nu+5)}{1.\ \ 2.\ \ 3\ldots\ldots\ldots(\nu-4)}\cos 8v +$$

$$\frac{2\nu(2\nu-1)(2\nu-2)\ldots\ldots(\nu+6)}{1.\ \ 2.\ \ 3\ \ldots\ldots\ (\nu-5)}\cos 10v +$$

&c.

Sin autem sit n numerus impar, erit

$$1 \cos v = \cos v$$
$$4 \cos v^3 = 3 \cos v + \cos 3v$$
$$16 \cos v^5 = 10 \cos v + 5 \cos 3v + \cos 5v$$
$$64 \cos v^7 = 35 \cos v + 21 \cos 3v + 7 \cos 5v + \cos 7v$$
$$256 \cos v^9 = 126 \cos v + 84 \cos 3v + 36 \cos 5v + 9 \cos 7v + \cos 9v$$
$$1024 \cos v^{11} = 462 \cos v + 330 \cos 3v + 165 \cos 5v + 55 \cos 7v + 11 \cos 9v + \cos 11v$$

&c.

Generatim si $n = 2\nu - 1$, stabit

$$2^{2\nu-1} \cos v^{2\nu-1} = \frac{(2\nu-1)(2\nu-2)\ldots\ldots(\nu+1)}{1 \cdot \quad 2 \ldots\ldots\ldots\ldots(\nu-1)} \cos v$$

$$+ \frac{(2\nu-1)(2\nu-2)\ldots\ldots\ldots(\nu+2)}{1 \cdot \quad 2 \ldots\ldots\ldots\ldots\ldots(\nu-2)} \cos 3v +$$

$$\frac{(2\nu-1)(2\nu-2)\ldots\ldots\ldots\ldots(\nu+3)}{1 \cdot \quad 2 \ldots\ldots\ldots\ldots\ldots\ldots(\nu-3)} \cos 5v +$$

$$\frac{(2\nu-1)(2\nu-2)\ldots\ldots\ldots\ldots(\nu+4)}{1 \cdot \quad 2 \ldots\ldots\ldots\ldots\ldots(\nu-4)} \cos 7v +$$

$$\frac{(2\nu-1)(2\nu-2)\ldots\ldots\ldots\ldots(\nu+5)}{1 \cdot \quad 2 \ldots\ldots\ldots\ldots\ldots(\nu-5)} \cos 9v +$$

$$\frac{(2\nu-1)(2\nu-2)\ldots\ldots\ldots\ldots(\nu+6)}{1 \cdot \quad 2 \ldots\ldots\ldots\ldots\ldots(\nu-6)} \cos 11v$$

&c.

651. Propositio VII. *Potestates quaslibet, quas proferunt sinus anguli cujuscumque, in simplices sinus, & cosinus convertere.*

Re-

Resolutio. Si n sit numerus integer (pro fractis enim incommoda provenit reductio), & pro v^n scribatur z^n, nec non pro valoribus potestatum v^n in Coroll. 2. Prop. VI. traditis valores in Prop. V. reperti substituantur, sequentes emergent reductiones

$$1\,\text{fin}\,z = \text{fin}\,z$$

$$2\,\text{fin}\,z^2 = -\cos 2z + \frac{1}{2}.2$$

$$4\,\text{fin}\,z^3 = -\text{fin}\,3z + 3\,\text{fin}\,z$$

(B)....

$$8\,\text{fin}\,z^4 = +\cos 4z - 4\cos 2z + \frac{1}{2}.6$$

$$16\,\text{fin}\,z^5 = +\text{fin}\,5z - 5\,\text{fin}\,3z + 10\,\text{fin}\,z$$

$$32\,\text{fin}\,z^6 = -\cos 6z + 6\cos 4z - 15\cos 2z + \frac{1}{2}20$$

$$64\,\text{fin}\,z^7 = -\text{fin}\,7z + 7\,\text{fin}\,5z - 21\,\text{fin}\,3z + 35\,\text{fin}\,z$$

$$128\,\text{fin}\,z^8 = +\cos 8z - 8\cos 6z + 28\cos 4z - 56\cos 2z + \frac{1}{2}70$$

&c.

Quod si n sit numerus negativus, facta consimili modo reductione formularum, quæ in Coroll. 3. Prop. VI. reperiuntur, prodibit

$$\frac{1}{2\,\sin z} = +\sin z + \sin 3z + \sin 5z + \sin 7z + \sin 9z + \&c.$$

$$\frac{1}{4\,\sin z^2} = -\cos 2z - 2\cos 4z - 3\cos 6z - 4\cos 8z - 5\cos 10z - \&c.$$

$$(C)\ldots\ \frac{1}{8\,\sin z^3} = -\sin 3z - 3\sin 5z - 6\sin 7z - 10\sin 9z - 15\sin 11z - \&c.$$

$$\frac{1}{16\,\sin z^4} = +\cos 4z + 4\cos 6z + 10\cos 8z + 20\cos 10z + 53\cos 12z + \&c.$$

$$\frac{1}{32\,\sin z^5} = +\sin 5z + 5\sin 7z + 15\sin 9z + 35\sin 11z + 70\sin 13z + \&c.$$

&c.

Ex ſedula hujuſmodi formularum (*B*), (*C*) inſpectione quadruplices f rmulæ generales eliciuntur, prout ipſius *z* exponens *n* fuerit numerus formæ vel $4m$, vel $4m-1$, vel $4m-2$, vel $4m-3$, proprietate progr ſſionis in ſingulis caſibus attentius conſiderata. Sic ſi ponatur $n = 4m$, quantitates $8\,\sin z^4$, $128\,\sin z^8$, &c. ſunt ſeligendæ, & formularum iſtas adæquantium (quæ circa coſinus verſant) inceſſu examinato, tandem colligitur

I.

I. $2^{4m-1} \sin z^{4m} = \cos 4mz - 4m \cos(4m-2)z$

$+ \frac{4m(4m-1)}{1 \cdot 2} \cos(4m-4)z -$

$\frac{4m(4m-1)(4m-2)}{1 \cdot 2 \cdot 3} \cos(4m-6)z +$

$+ \frac{1}{2} \cdot \frac{4m(4m-1)(4m-2)\ldots\ldots(2m+1)}{1 \cdot 2 \cdot 3 \ldots\ldots\ldots\ldots 2m}$

Pari modo si statuatur $n = 4m - 1$, captis formulis $4 \sin z^3$, $64 \sin z^7$, &c. quæ sinus intuentur, earumque expensa progressionis lege, invenietur

II. $2^{4m-2} \sin z^{4m-1} = - \sin(4m-1)z +$

$(4m-1) \sin(4m-3)z \frac{(4m-1)(4m-2)}{1 \cdot 2} \times$

$\sin(4m-5)z + \frac{(4m-1)(4m-2)(4m-3)\ldots(2m+2)}{1 \cdot 2 \cdot 3 \ldots\ldots (2m-1)}$

$\sin z$.

Ponendo autem $n = 4m - 2$, sumptisque formulis $2 \sin z^2$, $\sin z^6$, &c. quæ rursus ad cosinus referuntur, prodibit

III. $2^{4m-3} \sin z^{4m-2} = - \cos(4m-2)z +$

$(4m-2) \cos(4m-4)z - \frac{(4m-2)(4m-3)}{1 \cdot 2} \cos$

$(4m-$

$$(4m-6)z + \frac{(4m-2)(4m-3)(4m-4)}{1.\ 2.\ 3} \cos(4m-8)z$$

$$\ldots + \frac{1}{2}\,\frac{(4m-2)(4m-3)(4m-4)\ldots(2m)}{1.\ 2.\ 3\ \ldots\ (2m-1)}.$$

Si fiat $n = 4m - 3$, obveniunt formulæ $\sin z$, $16 \sin z^5$, &c. redeuntes ad finus, adeoque habemus

$$\text{IV. } 2^{4m-4} \sin z^{4m-3} = + \sin(4m-3)z - (4m-3)\sin(4m-5)z + \frac{(4m-3)(4m-4)}{1.\ 2} \sin(4m-7)z - \frac{(4m-3)(4m-4)(4m-5)}{1.\ 2.\ 3} \sin(4m-9)z$$

$$\ldots + \frac{(4m-3)(4m-4)(4m-5)\ldots(2m)}{1.\ 2.\ 3\ \ldots\ (2m-2)} \sin z.$$

Quod si n sit numerus negativus integer, nempe $n = -4m$, vel $= -(4m+1)$, vel $= -(4m+2)$, vel $= -(4m+3)$, quaternas pariter obtinebimus formulas generales sequentes

$$\text{I. } \frac{1}{2^{4m} \sin z^{4m}} = + \cos 4mz + 4m \cos(4m+2)z + \frac{4m(4m+1)}{1.\ 2} \cos(4m+4)z + \frac{4m(4m+1)(4m+2)}{1.\ 2.\ 3} \cos(4m+6)z + \&c.$$

II.

II. $$\frac{1}{2^{4m+1}\,\textit{sin}\,z^{4m+1}} = +\,\textit{sin}\,(4m+1)z + (4m+1)\,\textit{sin}\,(4m+3)z - \frac{(4m+2)\,(4m+3)}{1\,.\quad 2} \times \cos\,(4m+6)\,z - \frac{(4m+2)(4m+3)(4m+4)}{1\,.\quad 2\,.\quad 3} \cos(4m+8)z - \&c.$$

III. $$\frac{1}{2^{4m+2}\,\textit{sin}\,z^{4m+2}} = -\cos\,(4m+2)z - (4m+2)\cos\,(4m+4)z - \frac{(4m+2)\,(4m+3)}{1\,.\quad 2}\cos(4m+6)z - \frac{(4m+2)(4m+3)(4m+4)}{1\,.\quad 2\,.\quad 3}\cos(4m+8)z - \&c.$$

IV. $$\frac{1}{2^{4m-3}\,\textit{sin}\,z^{4m-3}} = -\,\textit{sin}\,(4m+3)z - (4m+3)\,\textit{sin}\,(4m+5)z - \frac{(4m+3)\,(4m+4)}{1\,.\quad 2}\,\textit{sin}\,(4m+7)z -$$

$$-\frac{(4m+3)(4m+4)(4m+5)}{1.\quad 2.\quad 3}$$

$$\sin(4m+9)z - \&c.$$

Quoties ergo n est numerus par, sive positivus, sive negativus, potestas $\sin z^n$ resolvitur in cosinus simplices angulorum multiplorum ipsius z. Sin autem n fuerit numerus impar, potestas $\sin us\ z^n$ resolvitur in sinus simplices angulorum multiplorum ipsius z. *Quod erat* &c.

652. *Cor.* 1. Si ponatur n numerus integer positivus par, nimirum ipsius $\sin z$ exponens n nunc $=4m$, nunc $=4m-2$, & formularum generalium exhibitarum ad hanc hypothesim pertinentium termini ordine inverso disponantur, ex duabus hujusmodi formulis una formabitur, videlicet

$$2\sin z^2 = 1 - \cos 2z$$

$$8\sin z^4 = 3 - 4\cos 2z + \cos 4z$$

$$32\sin z^6 = 10 - 15\cos 2z + 6\cos 4z - \cos 6z$$

$$128\sin z^8 = 35 - 56\cos 2z + 28\cos 4z - 8\cos 6z + \cos 8z$$

$$512\sin z^{10} = 126 - 210\cos 2z + 120\cos 4z - 45\cos 6z + 10\cos 8z - \cos 10z$$

$$2048\sin z^{12} = 462 - 792\cos 2z + 495\cos 4z - 220\cos 6z + 66\cos 8z - 12\cos 10z + \cos 12z$$

&c.

generatim vero

2^{2v-1}

$$2^{2\nu-1}\sin v^{2\nu} = \frac{1}{2}\cdot\frac{2\nu(2\nu-1)(2\nu-2)\ldots\ldots(\nu+1)}{1.\quad 2.\quad 3\ldots\ldots\ldots\ldots\nu}$$

$$\cos 2z - \frac{2\nu(2\nu-1)(2\nu-2)\ldots\ldots(\nu+2)}{1.\quad 2.\quad 3\ldots\ldots\ldots(\nu-1)}\cos 2z +$$

$$\frac{2\nu(2\nu-1)(2\nu-2)\ldots\ldots\ldots(\nu+3)}{1.\quad 2.\quad 3\ldots\ldots\ldots\ldots(\nu-2)}\cos 4z -$$

$$\frac{2\nu(2\nu-1)(2\nu-2)\ldots\ldots(\nu+4)}{1.\quad 2.\quad 3\ldots\ldots\ldots\ldots(\nu-3)}\cos 6z +$$

$$\frac{2\nu(2\nu-1)(2\nu-2)\ldots\ldots\ldots(\nu+5)}{1.\quad 2.\quad 3\ldots\ldots\ldots\ldots(\nu-4)}\cos 8z -$$

$$\frac{2\nu(2\nu-1)(2\nu-2)\ldots\ldots\ldots(\nu+6)}{1.\quad 2.\quad 3\qquad\qquad(\nu-5)}\cos 10z$$

&c.

653. *Coroll.* 2. Si exponens n fit numerus integer impar, duæ memoratæ formulæ generales in unam coalefcent, quæ erit

$1\sin z = \sin z$

$4\sin z^3 = 3\sin z - \sin 3z$

$16\sin z^5 = 10\sin z - 5\sin 3z + \sin 5z$

$64\sin z^7 = 35\sin z - 21\sin 3z + 7\sin 5z - \sin 7z$

$256\sin z^9 = 126\sin z - 84\sin 3z + 36\sin 5z -$
$9\sin 7z + \sin 9z$

$1024\sin z^{11} = 462\sin z - 330\sin 3z + 165\sin 5z -$
$55\sin 7z + 11\sin 9z - \sin 11z$

&c.

generaliter autem

$2^{2\nu-2}$

$$2^{2\nu-2}\int\!in\, z^{2\nu-1} = \frac{(2\nu-1)(2\nu-2)\ldots\ldots(\nu+1)}{1.\quad 2\ldots\ldots\ldots(\nu-1)}$$

$$\int\!in\, z - \frac{(2\nu-1)(2\nu-2)\ldots\ldots(\nu+2)}{1.\quad 2\quad\ldots\quad(\nu-2)}\int\!in\; 3z +$$

$$\frac{(2\nu-1)(2\nu-2)\ldots\ldots\ldots(\nu+3)}{1.\quad 2\quad\ldots\quad(\nu-3)}\int\!in\; 5z -$$

$$\frac{(2\nu-1)(2\nu-2)\ldots\ldots\ldots(\nu+4)}{(\nu-4)}\int\!in\; 7z$$

$$\frac{(2\nu-1)(2\nu-2)\ldots\ldots\ldots(\nu+5)}{1.\quad 2\quad\ldots\quad(\nu-5)}\int\!in\; 9z -$$

$$\frac{(2\nu-1)(2\nu-2)\ldots\ldots\ldots(\nu+6)}{1.\quad 2\quad\ldots\quad(\nu-6)}\int\!in\; 11z$$

&c.

CAPUT II.

DE QUANTITATUM PER SERIES EVOLUTIONE.

654. PRopofitio I. *Datam quantitatem in feriem convertere.*

Refolutio. Si fuerit $\frac{X}{Z}$ in feriem convertenda,

I. Fiat $\frac{X}{Z} = A + Bx + Cx^2 + Dx^3 + \&c.$ *Erit* $X + 0 + 0 + 0 \ldots = Z[A + Bx + Cx^2 + Dx^3 + \&c.]$ dum-

[i]nmodo littera x, cujus functio est X, etiam in [r]eperiatur. Quare peracta in casibus particulari[b]us multiplicatione, termini homologi, seu homo[ge]nei invicem comparentur, ac singuli indeterminatarum A, B, C, D, &c. valores eliciantur.

II. Si quantitas in seriem convertenda sit integra potestas indeterminata rationalis aut irrationalis duobus saltem terminis constans, facta ut supra æquatione, sumantur hinc inde logarithmi, quibus in seriem resolutis (257.), factaque dicta comparatione, valores A, B, C, D, &c. definientur.

III. Si fractionis datæ denominator sit potestas, ut $\frac{a+bx}{(1-cx)^n}$, duobus modis calculus iniri potest; scilicet vel potestas $(1-cx)^n$ evolvatur, deinde ut in num. I. procedatur; vel logarithmi ut in num. II. in auxilium vocentur. Sic erit $l[a+bx = nl[1-cx]+l[A+Bx+Cx^2+Cx^3+$ &c.$]$; tum resolutis logarithmis in series, comparatisque terminis homologis, valores coëfficientium A, B, C, obtinebuntur.

IV. Si in altero æquationis membro existat incognita diversa ab illa, quæ in altero membro ad plures potestates elata appareat; ut secundæ incognitæ valor per primæ terminos habeatur, eadem methodo est procedendum.

V. Si primus denominatoris terminus non sit quantitas constans, velut $\frac{a+bx+cx^2+\text{\&c.}}{x^m(1-\alpha x-\beta x^2-\text{\&c.})}$, fiat

646.

$$2^n \cos v^n = p^n + \frac{n}{1} p^{n-1} q + \frac{n(n-1)}{1.\ 2} p^{n-2} q^2 + \frac{n(n-1)(n-2)}{1.\ 2.\ 3} p^{n-3} q^3 + \&c.$$

$$2^n \cos v^n = q^n + \frac{n}{1} q^{n-1} p + \frac{n(n-1)}{1.\ 2} q^{n-2} p^2 + \frac{n(n-1)(n-2)}{1.\ 2.\ 3} q^{n-3} p^3 + \&c.$$

$$2^{n+1} \cos v^n = p^n + q^n + \frac{n}{1}\left(p^{n-2} + q^{n-2}\right)pq + \frac{n(n-1)}{1.\ 2}\left(p^{n-4} + q^{n-4}\right)p^2 q^2 + \&c.$$

ubi facta per 2 divisione, ob $pq = 1$, prodibit

$$2^n \cos v^n = \frac{1}{2}\left(p^n + q^n\right) + \frac{n}{1}.\frac{1}{2}\left(p^{n-2} + q^{n-2}\right) + \frac{n(n-1)}{1.\ 2}.\frac{1}{2}\left(p^{n-4} + q^{n-4}\right) + \&c.$$

substitue $2 \cos nv$ pro æquali $p^n + q^n$, ac tandem emerget

(A).... $2^n \cos v^n = \cos nv + \frac{n}{1} \cos(n-2)v + \frac{n(n-1)}{1.\ 2} \cos(n-4)v + \frac{n(n-1)(n-2)}{1.\ 2.\ 3} \cos(n-6)v$ $+ \&c.$ *Quod erat &c.*

646. *Coroll.* 1. Sit jam $n = 1$; quoniam $\cos(n-n)v = \cos(m-n)v$ (214. n. 5.) & $\cos 0 v = 1$, erit $2 \cos v = \cos v + \cos v = 2 \cos v$, adeoque $\cos v = \cos v$. Sit $n = 2$; habetur $2^2 \cos v^2 = \cos 2v + 2 \cos 0 v + \cos 2v = 2 \cos 2v + 2$. Sit $n = 3$; fiet $2^3 \cos v^3 = \cos 3v + 3 \cos v + 3 \cos v + \cos 3v = 2 \cos 3v + 6 \cos v$. Sit $n = 4$; erit $2^4 \cos v^4 = \cos 4v + 4 \cos 2v + 6 \cos 0 v + 4 \cos 2v + \cos 4v = 2 \cos 4v + 8 \cos 2v + 6$; &c.

Ratio est, quia quadratum tres habet terminos, cubus quatuor, biquadratum, quinque, &c. quare tot termini in expressione generali (A) sunt in casibus particularibus accipiendi, quot indicat $n + 1$.

647. *Coroll.* 2. Posito igitur n numero integro positivo, factaque divisione per 2, habetur

$$1. \cos v = \cos v$$

$$2. \cos v^2 = \cos 2v + \frac{1}{2} . 2$$

$$4. \cos v^3 = \cos 3v + 3 \cos v$$

$$8. \cos v^4 = \cos 4v + 4 \cos 2v + \frac{1}{2} . 6$$

$$16. \cos v^5 = \cos 5v + 5 \cos 3v + 10 \cos v$$

$$32. \cos v^6 = \cos 6v + 6 \cos 4v + 15 \cos 2v + \frac{1}{2} . 20$$

$$64. \cos v^7 = \cos 7v + 7 \cos 5v + 21 \cos 3v + 35 \cos v$$

$$182. \cos v^8 = \cos 8v + 8 \cos 6v + 28 \cos 4v + 56 \cos 2v + \frac{1}{2} 70 \text{ \&c.}$$

648. *Coroll.* 3. Si vero n sit numerus negativus, series (A) evadit infinita sic

$$\frac{1}{2\cos v} = \cos v - \cos 3v + \cos 5v - \cos 7v + \cos 9v - \&c.$$

$$\frac{1}{4\cos v^2} = \cos 2v - 2\cos 4v + 3\cos 6v - 4\cos 8v - 5\cos 10v - 6\cos 12v + \&c.$$

$$\frac{1}{8\cos v^3} = \cos 3v - 3\cos 5v + 6\cos 7v - 10\cos 9v + 15\cos 11v - 21\cos 13v + \&c.$$

$$\frac{1}{16\cos v^4} = \cos 4v - 4\cos 6v - 10\cos 8v - 20\cos 10v + 35\cos 12v - 56\cos 14v + \&c.$$

&c.

649. *Coroll.* 4. Quod si n fuerit numerus fractus, proveniet

$$\sqrt{2\cos v} = \cos\frac{1}{2}v + \frac{1}{2}\cos\frac{3}{2}v - \frac{1\cdot 1}{2\cdot 4}\cos\frac{7}{2}v + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\cos\frac{11}{2}v - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}\cos\frac{15}{2}v + \&c.$$

$$\frac{1}{\sqrt{2\cos v}} = \cos\frac{1}{2}v - \frac{1}{2}\cos\frac{5}{2}v + \frac{1\cdot 3}{2\cdot 4}\cos\frac{9}{2}v - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cos\frac{13}{2}v + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cos\frac{17}{2}v - \&c.$$

ubi coëfficientes iidem sunt cum illis, qui in extractione radicis per formulam Newtonianam obtinentur.

650.

650. *Scholion.* Præſtat interdum formulas Corollarii ſecundi commodius ordine inverſo diſponere, ipſas in duas claſſes, prout n ſit par vel impar, diſtribuendo. Sic ſi n ſit numerus par, fient

$$2\cos v^2 = 1 + \cos 2v$$
$$8\cos v^4 = 3 + 4\cos 2v + \cos 4v$$
$$32\cos v^6 = 10 + 15\cos 2v + 6\cos 4v + \cos 6v$$
$$128\cos v^8 = 35 + 56\cos 2v + 28\cos 4v + 8\cos 6v + \cos 8v$$
$$512\cos v^{10} = 126 + 210\cos 2v + 120\cos 4v + 45\cos 6v + 10\cos 8v + \cos 10v$$
$$2048\cos v^{12} = 462 + 792\cos 2v + 495\cos 4v + 220\cos 6v + 66\cos 8v + 12\cos 10v + \cos 12v$$

&c.

Generatim ſi fuerit $n = 2\nu$, habebitur

$$2^{2\nu-1}\cos v^{2\nu} = \frac{1}{2}\cdot\frac{2\nu(2\nu-1)(2\nu-2)\ldots\ldots(\nu+1)}{1.\ 2.\ 3\ldots\ldots\ldots\ldots\nu}$$

$$+\frac{2\nu(2\nu-1)(2\nu-2)\ldots\ldots(\nu+2)}{1.\ 2.\ 3\ldots\ldots(\nu-1)}\cos 2v +$$

$$+\frac{2\nu(2\nu-1)(2\nu-2)\ldots\ldots(\nu+3)}{1.\ 2.\ 3\ldots\ldots\ldots(\nu-2)}\cos 4v +$$

$$\frac{2\nu(2\nu-1)(2\nu-2)\ldots\ldots(\nu+4)}{1.\ 2.\ 3\ldots\ldots(\nu-3)}\cos 6v +$$

$$+\frac{2\nu(2\nu-1)(2\nu-2)\ldots\ldots(\nu+5)}{1.\ 2.\ 3\ldots\ldots(\nu-4)}\cos 8v +$$

$$\frac{2\nu(2\nu-1)(2\nu-2)\ldots\ldots(\nu+6)}{1.\ 2.\ 3\ldots\ldots(\nu-5}\cos 10v +$$

&c.

Sin autem sit n numerus impar, erit

$$1 \cos v = \cos v$$

$$4 \cos v^3 = 3 \cos v + \cos 3v$$

$$16 \cos v^5 = 10 \cos v + 5 \cos 3v + \cos 5v$$

$$64 \cos v^7 = 35 \cos v + 21 \cos 3v + 7 \cos 5v + \cos 7v$$

$$256 \cos v^9 = 126 \cos v + 84 \cos 3v + 36 \cos 5v + 9 \cos 7v + \cos 9v$$

$$1024 \cos v^{11} = 462 \cos v + 330 \cos 3v + 165 \cos 5v + 55 \cos 7v + 11 \cos 9v + \cos 11v$$

&c.

Generatim si $n = 2\nu - 1$, stabit

$$2^{2\nu-1} \cos v^{2\nu-1} = \frac{(2\nu-1)(2\nu-2)\ldots\ldots(\nu+1)}{1.\quad 2\ldots\ldots\ldots(\nu-1)} \cos v$$

$$+ \frac{(2\nu-1)(2\nu-2)\ldots\ldots\ldots(\nu+2)}{1.\quad 2\ldots\ldots\ldots\ldots(\nu-2)} \cos 3v +$$

$$\frac{(2\nu-1)(2\nu-2)\ldots\ldots\ldots(\nu+3)}{1.\quad 2\ldots\ldots\ldots\ldots(\nu-3)} \cos 5v +$$

$$\frac{(2\nu-1)(2\nu-2)\ldots\ldots\ldots(\nu+4)}{1.\quad 2\ldots\ldots\ldots\ldots(\nu-4)} \cos 7v +$$

$$\frac{(2\nu-1)(2\nu-2)\ldots\ldots\ldots(\nu+5)}{1.\quad 2\ldots\ldots\ldots\ldots(\nu-5)} \cos 9v +$$

$$\frac{(2\nu-1)(2\nu-2)\ldots\ldots\ldots(\nu+6)}{1.\quad 2\ldots\ldots\ldots\ldots(\nu-6)} \cos 11v$$

&c.

651. Propositio VII. *Potestates quaslibet, quas proferunt sinus anguli cujuscumque, in simplices sinus, & cosinus convertere.*

Re-

Resolutio. Si n sit numerus integer (pro fractis enim incommoda provenit reductio), & pro v^n scribatur z^n, nec non pro valoribus potestatum v^n in Coroll. 2. Prop. VI. traditis valores in Prop. V. reperti substituantur, sequentes emergent reductiones

(B)....

$$1\, \sin z = \sin z$$

$$2\, \sin z^2 = -\cos 2z + \frac{1}{2} \cdot 2$$

$$4\, \sin z^3 = -\sin 3z + 3 \sin z$$

$$8\, \sin z^4 = +\cos 4z - 4 \cos 2z + \frac{1}{2} \cdot 6$$

$$16\, \sin z^5 = +\sin 5z - 5 \sin 3z + 10 \sin z$$

$$32\, \sin z^6 = -\cos 6z + 6 \cos 4z - 15 \cos 2z + \frac{1}{2}\, 20$$

$$64\, \sin z^7 = -\sin 7z + 7 \sin 5z - 21 \sin 3z + 35 \sin z$$

$$128\, \sin z^8 = +\cos 8z - 8 \cos 6z + 28 \cos 4z - 56 \cos 2z + \frac{1}{2}\, 70$$

&c.

Quod si n sit numerus negativus, facta consimili modo reductione formularum, quæ in Coroll. 3. Prop. VI. reperiuntur, prodibit

$$\frac{1}{2\,\sin z} = +\sin z + \sin 3z + \sin 5z + \sin 7z + \sin 9z + \&c.$$

$$\frac{1}{4\,\sin z^2} = -\cos 2z - 2\cos 4z - 3\cos 6z - 4\cos 8z - 5\cos 10z - \&c.$$

$$(C)\ldots\ \frac{1}{8\,\sin z^3} = -\sin 3z - 3\sin 5z - 6\sin 7z - 10\sin 9z - 15\sin 11z - \&c.$$

$$\frac{1}{16\,\sin z^4} = +\cos 4z + 4\cos 6z + 10\cos 8z + 20\cos 10z + 53\cos 12z + \&c.$$

$$\frac{1}{32\,\sin z^5} = +\sin 5z + 5\sin 7z + 15\sin 9z + 35\sin 11z + 70\sin 13z + \&c.$$

&c.

Ex ſedula hujuſmodi formularum (*B*), (*C*) inſpectione quadruplices f rmulæ generales eliciuntur, prout ipſius *z* exponens *n* fuerit numerus formæ vel $4m$, vel $4m-1$, vel $4m-2$, vel $4m-3$, proprietate progr ſſionis in ſingulis caſibus attentius conſiderata. Sic ſi ponatur $n = 4m$, quantitates $8\sin z^4$, $128\sin z^8$, &c. ſunt ſeligendæ, & formularum iſtas adæquantium (quæ circa coſinus verſant) inceſſu examinato, tandem colligitur

I.

$$\text{I.}\quad 2^{4m-1} \sin z^{4m} = \cos 4mz - 4m\cos(4m-2)z + \frac{4m(4m-1)}{1\,.\,2}\cos(4m-4)z - \frac{4m(4m-1)(4m-2)}{1\,.\,2\,.\,3}\cos(4m-6)z + \ldots + \frac{1}{2}\,.\,\frac{4m(4m-1)(4m-2)\ldots(2m+1)}{1\,.\,2\,.\,3\ldots\ldots 2m}$$

Pari modo si statuatur $n = 4m - 1$, captis formulis $4 \sin z^3$, $64 \sin z^7$, &c. quæ sinus intuentur, earumque expensa progressionis lege, invenietur

$$\text{II.}\quad 2^{4m-2} \sin z^{4m-1} = -\sin(4m-1)z + (4m-1)\sin(4m-3)z\,\frac{(4m-1)(4m-2)}{1\,.\,2} \times \sin(4m-5)z + \frac{(4m-1)(4m-2)(4m-3)\ldots(2m+2)}{1\,.\,2\,.\,3\ldots\ldots(2m-1)} \sin z.$$

Ponendo autem $n = 4m - 2$, sumptisque formulis $2 \sin z^2$, $\sin z^6$, &c. quæ rursus ad cosinus referuntur, prodibit

$$\text{III.}\quad 2^{4m-3} \sin z^{4m-2} = -\cos(4m-2)z + (4m-2)\cos(4m-4)z - \frac{(4m-2)(4m-3)}{1\,.\,2}\cos$$

(4m-

$$(4m-6)z + \frac{(4m-2)(4m-3)(4m-4)}{1 \cdot\ 2 \cdot\ 3} \cos(4m-8)z$$

$$\ldots + \frac{1}{2}\,\frac{(4m-2)(4m-3)(4m-4)\ldots(2m)}{1 \cdot\ 2 \cdot\ 3\ \ldots\ (2m-1)}.$$

Si fiat $n = 4m - 3$, obveniunt formulæ $\sin z$, $16 \sin z^5$, &c. redeuntes ad finus, adeoque habemus

$$\text{IV. } 2^{4m-4} \sin z^{4m-3} = + \sin(4m-3)z - (4m-3)\sin(4m-5)z + \frac{(4m-3)(4m-4)}{1 \cdot\ 2} \sin(4m-7)z - \frac{(4m-3)(4m-4)(4m-5)}{1 \cdot\ 2 \cdot\ 3} \sin(4m-9)z$$

$$\ldots + \frac{(4m-3)(4m-4)(4m-5)\ldots(2m)}{1 \cdot\ 2 \cdot\ 3 \ldots\ (2m-2)} \sin z.$$

Quod si n sit numerus negativus integer, nempe $n = -4m$, vel $= -(4m+1)$, vel $= -(4m+2)$, vel $= -(4m+3)$, quaternas pariter obtinebimus formulas generales sequentes

$$\text{I. } \frac{1}{2^{4m} \sin z^{4m}} = + \cos 4mz + 4m\cos(4m+2)z + \frac{4m(4m+1)}{1 \cdot\ 2} \cos(4m+4)z + \frac{4m(4m+1)(4m+2)}{1 \cdot\ 2 \cdot\ 3} \cos(4m+6)z + \&c.$$

II.

II. $\frac{1}{2^{4m+1} \sin z^{4m+1}} = + \sin(4m+1)z + (4m+1)\sin(4m+3)z - \frac{(4m+2)(4m+3)}{1.\ 2} \times \cos(4m+6)z - \frac{(4m+2)(4m+3)(4m+4)}{1.\ 2.\ 3} \cos(4m+8)z - \&c.$

III. $\frac{1}{2^{4m+2} \sin z^{4m+2}} = - \cos(4m+2)z - (4m+2)\cos(4m+4)z - \frac{(4m+2)(4m+3)}{1.\ 2} \cos(4m+6)z - \frac{(4m+2)(4m+3)(4m+4)}{1.\ 2.\ 3} \cos(4m+8)z - \&c.$

IV. $\frac{1}{2^{4m-3} \sin z^{4m-3}} = - \sin(4m+3)z - (4m+3)\sin(4m+5)z - \frac{(4m+3)(4m+4)}{1.\ 2} \sin(4m+7)z -$

$$-\frac{(4m+3)(4m+4)(4m+5)}{1.\quad 2.\quad 3}$$

$$\mathit{fin}\,(4m+9)\,z - \&c.$$

Quoties ergo n eſt numerus par, ſive poſitivus, ſive negativus, poteſtas $\mathit{fin}\, z^n$ reſolvitur in coſinus ſimplices angulorum multiplorum ipſius z. Sin autem n fuerit numerus impar, poteſtas $\mathit{finus}\, z^n$ reſolvitur in ſinus ſimplices angulorum multiplorum ipſius z. *Quod erat* &c.

652. *Cor.* 1. Si ponatur n numerus integer poſitivus par, nimirum ipſius $\mathit{fin}\, z$ exponens n nunc $=4m$, nunc $=4m-2$, & formularum generalium exhibitarum ad hanc hypotheſim pertinentium termini ordine inverſo diſponantur, ex duabus hujuſmodi formulis una formabitur, videlicet

$$2\,\mathit{fin}\, z^2 = 1 - \cos 2z$$

$$8\,\mathit{fin}\, z^4 = 3 - 4\cos 2z + \cos 4z$$

$$32\,\mathit{fin}\, z^6 = 10 - 15\cos 2z + 6\cos 4z - \cos 6z$$

$$128\,\mathit{fin}\, z^8 = 35 - 56\cos 2z + 28\cos 4z - 8\cos 6z + \cos 8z$$

$$512\,\mathit{fin}\, z^{10} = 126 - 210\cos 2z + 120\cos 4z - 45\cos 6z + 10\cos 8z - \cos 10z$$

$$2048\,\mathit{fin}\, z^{12} = 462 - 792\cos 2z + 495\cos 4z - 220\cos 6z + 66\cos 8z - 12\cos 10z + \cos 12z$$

&c.

generatim vero

$2^{2\nu-1}$

$$2^{2\nu-1} \mathit{fin}\, v^{2\nu} = \frac{1}{2} \cdot \frac{2\nu(2\nu-1)(2\nu-2)\ldots\ldots(\nu+1)}{1.\quad 2.\quad 3\ldots\ldots\ldots\nu}$$

$$\cos 2z - \frac{2\nu(2\nu-1)(2\nu-2)\ldots\ldots(\nu+2)}{1.\quad 2.\quad 3\ldots\ldots(\nu-1)} \cos 2z +$$

$$\frac{2\nu(2\nu-1)(2\nu-2)\ldots\ldots(\nu+3)}{1.\quad 2.\quad 3\ldots\ldots(\nu-2)} \cos 4z -$$

$$\frac{2\nu(2\nu-1)(2\nu-2)\ldots\ldots(\nu+4)}{1.\quad 2.\quad 3\ldots\ldots(\nu-3)} \cos 6z +$$

$$\frac{2\nu(2\nu-1)(2\nu-2)\ldots\ldots(\nu+5)}{1.\quad 2.\quad 3\ldots\ldots(\nu-4)} \cos 8z -$$

$$\frac{2\nu(2\nu-1)(2\nu-2)\ldots\ldots(\nu+6)}{1.\quad 2.\quad 3\quad (\nu-5)} \cos 10z$$

&c.

653. *Coroll.* 2. Si exponens n ſit numerus integer impar, duæ memoratæ formulæ generales in unam coaleſcent, quæ erit

$$1\, \mathit{fin}\, z = \mathit{fin}\, z$$
$$4\, \mathit{fin}\, z^3 = 3\, \mathit{fin}\, z - \mathit{fin}\, 3z$$
$$16\, \mathit{fin}\, z^5 = 10\, \mathit{fin}\, z - 5\, \mathit{fin}\, 3z + \mathit{fin}\, 5z$$
$$64\, \mathit{fin}\, z^7 = 35\, \mathit{fin}\, z - 21\, \mathit{fin}\, 3z + 7\, \mathit{fin}\, 5z - \mathit{fin}\, 7z$$
$$256\, \mathit{fin}\, z^9 = 126\, \mathit{fin}\, z - 84\, \mathit{fin}\, 3z + 36\, \mathit{fin}\, 5z - 9\, \mathit{fin}\, 7z + \mathit{fin}\, 9z$$
$$1024\, \mathit{fin}\, z^{11} = 462\, \mathit{fin}\, z - 330\, \mathit{fin}\, 3z + 165\, \mathit{fin}\, 5z - 55\, \mathit{fin}\, 7z + 11\, \mathit{fin}\, 9z - \mathit{fin}\, 11z$$

&c.

generaliter autem

$2^{2\nu-2}$

$$2^{2\nu-2}\sin z^{2\nu-1} = \frac{(2\nu-1)(2\nu-2)\ldots\ldots(\nu+1)}{1.\quad 2\ldots\ldots\ldots(\nu-1)}$$

$$\sin z - \frac{(2\nu-1)(2\nu-2)\ldots\ldots(\nu+2)}{1.\quad 2\qquad\qquad(\nu-2)}\sin 3z +$$

$$\frac{(2\nu-1)(2\nu-2)\ldots\ldots\ldots(\nu+3)}{1.\quad 2\qquad\qquad(\nu-3)}\sin 5z -$$

$$\frac{(2\nu-1)(2\nu-2)\ldots\ldots\ldots(\nu+4)}{(\nu-4)}\sin 7z$$

$$\frac{(2\nu-1)(2\nu-2)\ldots\ldots\ldots(\nu+5)}{1.\quad 2\qquad\qquad(\nu-5)}\sin 9z -$$

$$\frac{(2\nu-1)(2\nu-2)\ldots\ldots\ldots(\nu+6)}{1.\quad 2\qquad\qquad(\nu-6)}\sin 11z$$

&c.

CAPUT II.

DE QUANTITATUM PER SERIES EVOLUTIONE.

654. PRopofitio I. *Datam quantitatem in feriem convertere.*

Refolutio. Si fuerit $\frac{X}{Z}$ in feriem convertenda,

I. Fiat $\frac{X}{Z} = A + Bx + Cx^2 + Dx^3 + \&c.$ *Erit* $X + 0 + 0 + 0 \ldots = Z[A + Bx + Cx^2 + Dx^3 + \&c.]$ dum-

dummodo littera x, cujus functio est X, etiam in Z reperiatur. Quare peracta in casibus particularibus multiplicatione, termini homologi, seu homogenei invicem comparentur, ac singuli indeterminatarum A, B, C, D, &c. valores eliciantur.

II. Si quantitas in seriem convertenda sit integra potestas indeterminata rationalis aut irrationalis duobus saltem terminis constans, facta ut supra æquatione, sumantur hinc inde logarithmi, quibus in seriem resolutis (257.), factaque dicta comparatione, valores A, B, C, D, &c. definientur.

III. Si fractionis datæ denominator sit potestas, ut $\frac{a+bx}{(1-cx)^n}$, duobus modis calculus iniri potest; scilicet vel potestas $(1-cx)^n$ evolvatur, deinde ut in num. I. procedatur; vel logarithmi ut in num. II. in auxilium vocentur. Sic erit $l[a+bx = nl[1-cx] + l[A+Bx+Cx^2+Cx^3+\&c.]$; tum resolutis logarithmis in series, comparatisque terminis homologis, valores coëfficientium A, B, C, obtinebuntur.

IV. Si in altero æquationis membro existat incognita diversa ab illa, quæ in altero membro ad plures potestates elata appareat; ut secundæ incognitæ valor per primæ terminos habeatur, eadem methodo est procedendum.

V. Si primus denominatoris terminus non sit quantitas constans, velut $\frac{a+bx+cx^2+\&c.}{x^m(1-\alpha x-\beta x^2-\&c.)}$, fiat

fiat $\frac{a + bx - cx^2 + \&c.}{1 - \alpha x - \beta x^2 - \&c.} = A + Bx + Cx^2 + Dx^3 +$ &c. neglecto denominatoris factore x^m; atque tunc erit $\frac{a + bx + cx^2 + \&c.}{x^m (1 - \alpha x - \beta x^2 - \&c.)} = \frac{A}{x^m} + \frac{B}{x^{m-1}} + \frac{C}{x^{m-2}} + \&c.$

VI. Si detur quantitas radicalis in seriem convertenda, facta prius æquatione, ut in num. I. elevetur utrumque æquationis membrum ad potestatem indicatam ab exponente ejusdem radicalis, deinde terminis homogeneis ut supra invicem collatis, quæsiti indeterminatarum A, B, C, D, &c. valores emergent, per quam methodum potestatum perfectarum radices innotescent.

Hæc omnia per Exempla luculentius declarabuntur.

Exemplum I. Sit fractio $\frac{a}{b + x}$ in seriem redigenda. Pono $\frac{a}{b + x} = A + Bx + Cx^2 + Dx^3 + \&c.$ Multiplicando igitur per $b + x$, stabit $a + o + o + o \ldots\ldots$

$$= Ab + Bbx + Cbx^2 + Dbx^3 + \&c.$$
$$+ Ax + Bx^2 + Cx^3 + \&c.$$

Fiat terminorum homogeneorum comparatio, eritque $Ab = a$; ergo $A = \frac{a}{b}$; $Bb + A = o$, ergo $B = -\frac{a}{b^2}$;

$Cb +$

$Cb + B = 0$, ergo $C = \frac{a}{b^3}$; $Db + C = 0$; ergo $D = -\frac{a}{b^4}$; &c. quibus valoribus ſubſtitutis prodit $\frac{a}{b+x} = \frac{a}{b} - \frac{ax}{b^2} + \frac{ax^2}{b^3} - \frac{ax^3}{b^4} +$ &c. prorſus ut alibi (46.).

655. *Coroll.* 1. Si fiat $x = cz$; erit $\frac{a}{b+cz} = \frac{a}{b} - \frac{acz}{b^2} + \frac{ac^2z^2}{b^3} - \frac{ac^3z^3}{b^4} +$ &c.

656. *Coroll.* 2. Si ſtatuatur $a = 1, b = 1$; erit $\frac{1}{1+x} = 1 - x + x^2 - x^3 +$ &c. Quod ſi fuerit x quantitas negativa, erit $\frac{1}{1-x} = 1 + x + x^2 + x^3 +$ &c. Si in locum x ſufficias x^n, fiet $\frac{1}{1 \pm x^n} = 1 \mp x^n + x^{2n} \mp x^{3n} + x^{4n} \mp$ &c.

657. *Coroll.* 3. Si fuerit $b = 0$, foret $A = \frac{a}{0}$,

atque ideo omnes ſeriei termini evaderent infiniti. Hinc numeri V. ratio eluceſcit; nam ut primus terminus A innoteſcat, fieri debet ejuſdem cum quantitate conſtanti comparatio.

Exemplum II. Sit $\frac{a+bx}{f+gx+hx^2}$. Pono $a+bx=$

$(f+gx+hx^2)(A+Bx+Cx^2+Dx^3+$ &c.$)$, ſive

$$a+bx+0+0\ldots = Af+Bfx+Cfx^2+Dfx^3+Efx^4+ \text{\&c.}$$
$$+Agx+Bgx^2+Cgx^3+Dgx^4+ \text{\&c.}$$
$$+Ahx^2+Bhx^3+Chx^4+ \text{\&c.}$$

Itaque facta comparatione, erit $Af=a$, ſeu $A=\frac{a}{f}$; $Bf+Ag=b$, *ſeu* $B=\frac{b}{f}-\frac{ag}{f^2}$. Præterea

$$Cf+Bg+Ah=0$$
$$Df+Cg+Bh=0$$
$$Ef+Dg+Ch=0$$
&c.

Datis igitur duobus coëfficientibus contiguis B, A proxime ſubſequens C invenitur; datis C, B, invenitur D. Generatim datis Q, P, invenitur R, litteris f, g, h eundem ſemper locum, prout extant in denominatore, ſervantibus.

658. *Coroll.* I. Ergo ſi $a=1$, $b=2$, $f=1$, $g=-1$, $h=-1$, erit

$$\left.\begin{array}{l} C-B-A=0 \\ D-C-B=0 \\ E-D-C=0 \\ \text{\&c.} \end{array}\right\}\ \text{ſeu}\ \left\{\begin{array}{l} C=B+A \\ D=C+B \\ E=D+C \\ \text{\&c.} \end{array}\right.$$

&

& fractio $\frac{1+2x}{1-x-xx} = 1 + 3x + 4x^2 + 7x^3 + 11x^4 + 18x^5 +$ &c.

659. *Coroll.* 2. Si quatuor fuerint in denominatoris terminis coëfficientes, nempe f, g, h, i, tunc terni coëfficientes contigui proxime subsequentem constituent; etenim erit $Df + Cg + Bh + Ai = 0$; $Ef + Dg + Ch + Bi = 0$; &c. & sic deinceps.

660. *Coroll.* 3. Ut autem valorum, quos habent litteræ A, B, C, &c. pro numero terminorum in numeratore existentium progressio in conspectum veniat, esto generatim $\frac{a + bx + cx^2 + dx^3 + \&c.}{1 - \alpha x - \beta x^2 - \gamma x^3 - \delta x^4 - \&c.}$ $= A + Bx + Cx^2 + Dx^3 +$ &c. Institutis de more operationibus, proveniet

$A = a$

$B = \alpha A + b$

$C = \alpha B + \beta A + c$

$D = \alpha C + \beta B + \gamma A + d$

$E = \alpha D + \beta C + \gamma B + \delta A + e$

&c.

Exhaustis igitur numeratoris (dummodo hic in infinitum non excurrat) terminis, reliqui litterarum majuscularum valores legem constantem ex aliquot præcedentibus, prout in Coroll. antec. indicavimus, servabunt.

661. *Scholion.* Series universæ, quæ sic oriuntur, *recurrentes* appellantur, quia ut termini cujuslibet subsequentis coëfficiens obtineatur, ad aliquot ex

antecedentibus est recurrendum. Istarum serierum ordo, seu gradus designatur ex numero fixo terminorum præcedentium, qui ad earum terminos quoscumque continuandos vel soli, vel per quantitates constantes multiplicati accipiuntur. Si enim quilibet seriei terminus ab unico termino præcedente oriatur, series erit primi ordinis; si a duobus terminis, series erit secundi ordinis, & ita porro. Generatim si series a fractione nascatur, fractionis hujus denominator ordinem seriei recurrentis hinc genitæ per maximum incognitæ exponentem exhibebit. Sic in serie supra exhibita $1 + 3x + 4x^2 + 7x^3 + 11x^4$ $+$ &c. quilibet coëfficiens habetur duos proxime antecedentes addendo, adeoque est series secundi ordinis. Cavendum autem, ne fractio proposita sit spuria, secus lex ista turbaretur, ut accideret sumendo

$\frac{1 + 2x - x^3}{1 - x - xx} = A + Bx + Cx^2 +$ &c.; prodiret enim

series $1 + 3x + 4x^2 + 6x^3 + 10x^4 + 16x^5 +$ &c. ubi terminus quartus $6x^3$ a sancita lege deflectit. Quantitates autem, quibus termini antecedentes multiplicantur, ut terminus proxime subsequens in dictis seriebus recurrentibus comperiatur, sub *scalarum relationis* nomine veniunt.

Exemplum III. Sit $(a + x)^n$. Ergo $a^n \left(1 + \frac{x}{a}\right)^n$

$= A + Bx + Cx^2 +$ &c. sumptisque logarithmis,

provenit $nla + nl\left(1 + \frac{x}{a}\right) = lA + l\left(1 + \frac{Bx}{A} + \right.$

Cx^2

$\frac{Cx^2}{A} + \frac{Dx^3}{A} +$ &c.) quibus resolutis in series, exurgit

$$\frac{nx}{a} - \frac{nx^2}{2a^2} + \frac{nx^3}{3a^3} - \frac{nx^4}{4a^4} + \&c. = \frac{Bx}{A}$$

$$+ \frac{Cx^2}{A} + \frac{Dx^3}{A} + \&c.$$

$$- \frac{B^2x^2}{2A^2} - \frac{2BCx^3}{2A^2}$$

$$+ \frac{B^3x^3}{3A^3}$$

Comparatis igitur terminis homologis, ob $nla = lA$, ut ex secunda æquatione patet, habetur

$$A = a^n$$

$$B = \frac{An}{a} = \frac{n}{1} a^{n-1}$$

$$C = \frac{B^2}{2A} - \frac{An}{2a^2} = \frac{n^2 a^{n-2} - na^{n-2}}{1 \,.\, 2} = \frac{n \,.\, \overline{n-1}}{1 \,.\, 2} a^{n-2}$$

$$D = \frac{nA}{3a^3} + \frac{BC}{A} - \frac{B^3}{3A^3} = \frac{2na^{n-3}}{6} + \frac{(3n^3 - 3n^2 - 2n^3)}{6}$$

$$= \frac{n(n-1)(n-2)}{1 \,.\, 2 \,.\, 3} a^{n-3} \text{ \&c.}$$

quare valoribus hisce in serie $A + Bx + Cx^2 + Dx^3 +$ &c. substitutis, prodit rursus, (& quidem directe) formula Newtoniana pro binomio ad quamcumque dignitatem attollendo.

662. *Coroll.* Ergo quicumque valor positivus aut negativus, integer aut fractus, realis aut imaginarius indeterminatæ n adsignari volupe sit, semper (repetito ad normam valoris adsumpti calculo) constabit, formulam Newtonianam omnibus casibus, nullo excepto, satisfacere posse; adeoque simplicissima nostra reductio quantitatis $(a + b\sqrt{-1})^{m + n\sqrt{-1}}$ ad expressionem $A + B\sqrt{-1}$ remanet perfecte demonstrata (132.).

663. *Scholion.* Formulæ generalis enunciatæ inventio, quæ summo Newtono communiter adscribitur, a celeberrimo Joh. Bernoullio (Oper. omn. T. IV. *Remarques sur le Calcul Integral de Monsieur* Stone, *pag.* 173.) Pascalio tribuitur his verbis: „ Nous avons trouvé ce merveilleux Theoreme, „ aussi bien que Monsieur Newton, d'une maniere „ plus simple que la sienne. Feu Monsieur Pascal a „ été le premier qui l'a inventé ".

Exemplum IV. Finitinomium, vel infinitinomium $a + bx + cx^2 + dx^3 +$ &c. ad dignitatem m perfectam aut imperfectam erigere.

Ergo $(a + bx + cx^2 + dx^3 + \&c.)^m = A + Bx + Cx^2 + Dx^3 +$ &c. sumptisque logarithmis, erit $mla +$

$$ml\left(1 + \frac{b}{a}x + \frac{c}{a}x^2 + \frac{d}{a}x^3 + \&c.\right) = lA +$$

l

$l(1+\frac{B}{A}x+\frac{C}{A}x^2+\frac{D}{A}x^3+\&c.)$. Logarithmi in series resoluti præbent $\frac{mbx}{a}+\frac{mcx^2}{a}+\frac{mdx^3}{a}+\&c. =$

$$-\frac{mb^2x^2}{2a^2}-\frac{2mbcx^3}{2a^2}$$

$$+\frac{mb^3x^3}{3a^3}$$

$$\frac{B}{A}x+\frac{C}{A}x^2+\frac{D}{A}x^3+\&c.$$

$$-\frac{B^2}{2A^2}x^2-\frac{2BC}{2A^2}x^3$$

$$+\frac{B^3x^3}{3A^3}$$

que homologis inter se collatis, invenietur

$$\frac{Ab}{a}=\frac{m}{1}a^{m-1}b$$

$$C=\frac{mAc}{a}-\frac{mAb^2}{2a^2}+\frac{B^2}{2A}=\frac{m}{1}a^{m-1}c+$$

$$\frac{m(m-1)}{1.\ 2}a^{m-2}b^2$$

V 4 D

$$D = \frac{mAd}{a} - \frac{2mAbc}{2a^2} + \frac{mAb^3}{3a^3} + \frac{2BC}{2A} - \frac{B^3}{3A^2}$$

$$= \frac{m}{1} a^{m-1} d + \frac{m(m-1)}{1 \cdot 2} a^{m-2} \times 2bc +$$

$$\frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} a^{m-3} b^3$$

&c.

Sic prosequendo, reliqui litterarum E, F, G, &c. valores reperientur, donec libuerit. Quocirca factis ut in Exemplo antecedente substitutionibus, si quantitas data sit infinitinomium, redibit formula generalis jam exhibita (70.) pro infinitinomii potestati, si m est numerus integer, vel radice qualibet, si m est numerus fractus, obtinenda.

664. *Coroll.* 1. Quoniam $(1 + \alpha x)^{m-1} =$

$$+ \frac{m-1}{1} \alpha x + \frac{(m-1)(m-2)}{1 \cdot 2} \alpha^2 x^2 +$$

$\frac{(m-1)(m-2)(m-3)}{1 \cdot 2 \cdot 3} \alpha^3 x^3 +$ &c. omnes termini ex primo, qui est 1, definiri poterunt, si posita de more hac serie $= 1 + Ax + Bx^2 + Cx^3 \ldots +$

$Lx^{n-2} + Mx^{n-1} + Nx^n +$ &c. quilibet coëfficiens N ita ex proxime antecedente M determinetur, ut

sit $N = \frac{m-n}{n} \alpha M$, ubi n exprimat ipsius x exponen

nentem. Sic facto $n = 1$ (qui ipsius x est exponens in secundo seriei termino), quum sit primus terminus $M = 1$, erit N, sive $A = \frac{m-1}{1}\alpha$; deinde posito $n = 2$, erit $N = \frac{m-2}{2}\alpha M$, sive sumptis B pro N, & A pro M, erit $B = \frac{m-2}{2}\alpha A$, nimirum $B = \frac{(m-1)(m-2)}{1 \cdot 2}\alpha^2$. Pari ratione habetur $C = \frac{m-3}{3}\alpha B = \frac{(m-1)(m-2)(m-3)}{1 \cdot 2 \cdot 3}\alpha^3$, & ita porro, quod cum allata serie conspirat.

665. *Coroll.* 2. Similiter $(1 + \alpha x + \beta xx)^{m-1} = 1 + \frac{m-1}{1}(\alpha x + \beta xx) + \frac{(m-1)(m-2)}{1 \cdot 2}(\alpha x + \beta xx)^2 + \&c.$

$$= 1 + \frac{m-1}{1}\alpha x + \frac{(m-1)(m-2)}{1 \cdot 2}\alpha^2 x^2 + \frac{(m-1)(m-2)(m-3)}{1 \cdot 2 \cdot 3}\alpha^3 x^3 + \&c.$$

$$+ \frac{m-1}{1}\beta x^2 + \frac{(m-1)(m-2)}{1 \cdot 2} 2\alpha\beta x^3 + \&c.$$ terminis secundum ipsius x potestates dispositis. In

In hac ſerie termini omnes ex primo definiuntur, quemlibet coëfficientem ex duobus proxime antecedentibus derivando; nempe ſi tres coëfficientes ſe ſtatim ſubſequentes ſint L, M, N, erit $N = \frac{m-n}{n}\alpha M + \frac{2m-n}{n}\beta L$. Erit igitur

$$A = \frac{m-1}{1}\alpha$$

$$B = \frac{m-2}{2}\alpha A + \frac{2m-2}{2}\beta$$

$$C = \frac{m-3}{3}\alpha B + \frac{2m-3}{3}\beta A$$

$$D = \frac{m-4}{4}\alpha C + \frac{2m-4}{4}\beta B.$$

&c.

666. *Corollarium* 3. Ita proſequendo, ſi fuerit $(1+\alpha x+\beta x^2+\gamma x^3)^{m-1}$; ex tribus terminis antecedentibus quilibet proxime ſubſequens coëfficiens ſimili modo determinabitur per formulam $N = \frac{m-n}{n}\alpha M + \frac{2m-n}{n}\beta L + \frac{3m-n}{n}\gamma K$, atque obtinebitur

$A =$

$$A = \frac{m-1}{1}\alpha$$

$$B = \frac{m-2}{2}\alpha A + \frac{2m-2}{2}\beta$$

$$C = \frac{m-3}{3}\alpha B + \frac{2m-3}{3}\beta A + \frac{3m-3}{3}\gamma$$

$$D = \frac{m-4}{4}\alpha C + \frac{2m-4}{4}\beta B + \frac{3m-4}{4}\gamma A$$

$$E = \frac{m-5}{5}\alpha D + \frac{2m-5}{5}\beta C + \frac{3m-5}{5}\gamma B$$

&c.

667. *Corollarium* 4. Ergo generatim ſi detur $(1 + \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \&c.)^{m-1}$, norma ex ſuperioribus petita, ſinguli coëfficientes ſic ex proxime antecedentibus per inductionem definientur

$$A = \frac{m-1}{1}\alpha$$

$$B = \frac{m-2}{2}\alpha A + \frac{2m-2}{2}\beta$$

$$C = \frac{m-3}{3}\alpha B + \frac{2m-3}{3}\beta A + \frac{3m-3}{3}\gamma$$

D =

$$D = \frac{m-4}{4}\alpha C + \frac{2m-4}{4}\beta B + \frac{3m-4}{4}\gamma A + \frac{4m-4}{4}\delta$$

$$E = \frac{m-5}{5}\alpha D + \frac{2m-5}{5}\beta C + \frac{3m-5}{5}\gamma B + \frac{4m-5}{5}\delta A + \frac{5m-5}{5}\varepsilon$$

&c.

Quod si proponatur $\frac{1}{(1-\alpha x-\beta x^2-\gamma x^3-\delta x^4-\&c.)^{m+1}}$,

sive $(1-\alpha x-\beta x^2-\gamma x^3-\delta x^4-\&c.)^{-m-1}$; tam m, quam litteræ α, β, γ, δ, &c. in adnexo paradigmate negative accipiantur; formula enim generalis pro quolibet termino ex proxime præcedentibus eliciendo tunc erit $N = \frac{m+n}{n}\alpha M + \frac{2m+n}{n}\beta L + \frac{3m+n}{n}\gamma K + \frac{4m+n}{n}\delta I + \&c. \ldots\ldots + \frac{pm+n}{n}\omega$.

Exemplum V. Sit $x = ay + by^2 + cy^3 + dy^4 + \&c.$ inveniendus est valor y in terminis potentiarum ipsius x.

Sta-

Statue $y = Ax + Bx^2 + Cx^3 + Dx^4 + \&c.$ quo valore in locum ipsius y in proposita suffecto, habetur

$$\left.\begin{array}{llll} ay = Aax + Bax^2 + Cax^3 & + Dax^4 & + \&c. \\ by^2 = \ldots\ldots A^2bx^2 + 2ABbx^3 & + B^2bx^4 & + \&c. \\ & + 2ACbx^4 & \\ cy^3 = \ldots\ldots\ldots\ldots + A^3cx^3 & + 3A^2Bcx^4 & + \&c. \\ dy^4 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots & + A^4dx^4 & + \&c. \end{array}\right\} = 0$$

factaque solita terminorum homogeneorum comparatione, reperitur $Aa = 1$, adeoque $A = \frac{1}{a}$; $Ba + A^2b = 0$, unde $B = -\frac{b}{a^3}$; $Ca + 2ABb + A^3c = 0$, quare $C = \frac{2b^2 - ac}{a^5}$; $Da + B^2b + 2ACb + 3A^2Bc + A^4d = 0$; ac proinde $D = \frac{5abc - a^2d - 5b^3}{a^7}$; &c. Ergo

$$y = \frac{x}{a} - \frac{bx^2}{a^3} + \frac{2b^2 - ac}{a^5}x^3 + \frac{5abc - a^2d - 5b^3}{a^7}x^4 + \&c.$$

668. *Coroll.* 1. Si fuerit $a = 1$, $b = \frac{1}{2}$, $c = \frac{1}{3}$, $d = \frac{1}{4}$ &c. fiet $y = \frac{x}{1} - \frac{x^2}{1.2} + \frac{x^3}{1.2.3} - \frac{x^4}{1.2.3.4} + \&c.$

Vi-

Viciſſim hac poſita æquatione, ſtabit

$$u = \frac{y}{1} + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \text{\&c.}$$

Quod ſi proponatur $u = \frac{y}{a} - \frac{y^2}{2a^2} + \frac{y^3}{3a^3} - \frac{y^4}{4a^4} + \text{\&c.}$, invenietur $\frac{y}{a} = \frac{u}{1} + \frac{u^2}{1.2} + \frac{u^3}{1.2.3} + \frac{u^4}{1.2.3.4} + \text{\&c.}$ & viciſſim &c.

669. *Coroll.* 2. Si primus datæ ſeriei terminus fuerit conſtans, velut $z = a + by + cy^2 + dy^3 + \text{\&c.}$ fiat $z - a = u$, eritque $u = by + cy^2 + dy^3 + \text{\&c.}$ tum operationes, ut ſupra inſtitui poſſunt.

670. *Scholion.* Si data æquatione $(1 + a)^u = 1 + y$, adſignari debeat ipſius y valor in terminis potentiarum alterius u; expedit evolvere $(1 + a)^u$ per Exemplum III. ex quo fiet $1 + \frac{u}{1} a + \frac{u(u-1)}{1.2} a^2 + \frac{u(u-1)(u-2)}{1.2.3} a^3 + \text{\&c.} = 1 + y$; adeoque

$$y = \frac{u}{1} a + \frac{u(u-1)}{1.2} a^2 + \frac{u(u-1)(u-2)}{1.2.3} a^3 + \text{\&c.}$$

Exem-

Exemplum VI. Sit generaliter

$$x = \alpha y^m + \beta y^{m+n} + \gamma y^{m+2n} + \delta y^{m+3n} + \&c.$$

fiat $v = \frac{x}{\alpha}$, eritque $v = y^m + \frac{\beta}{\alpha} y^{m+n} + \frac{\gamma}{\alpha} y^{m+2n} + \&c.$

Ponatur $y = v^{\frac{1}{m}} + Bv^{\frac{1+n}{m}} + Cv^{\frac{1+2n}{m}} + Dv^{\frac{1+3n}{m}} + \&c.$

Ut habeantur per v poteſtates ipſius y, ſumpta formula expoſita in Exemplo IV. pro polynomii poteſtate indeterminata invenienda, ſubſtitutiſque $v^{\frac{1}{m}}$ pro a; B pro b; $v^{\frac{1+n}{m}}$ pro x; C pro c; $v^{\frac{1+2n}{m}}$ pro x^2, &c. proveniet

$$y^m = v + mBv^{\frac{m+n}{m}} + mCv^{\frac{m+2n}{m}} \ \&c.$$
$$+ \frac{m(m-1)}{2} B^2 v^{\frac{m+2n}{m}}$$

$$\frac{\beta}{\alpha} y^{m+n} = \frac{\beta}{\alpha} v^{\frac{m+n}{m}} + (m+n) Bv^{\frac{m+2n}{m}} \ \&c.$$

$$\frac{\gamma}{\alpha} y^{m+2n} = \ldots\ldots\ldots\ldots \frac{\gamma}{\alpha} v^{\frac{m+2n}{m}} \ \&c.$$

Conferantur termini homologi, eritque $mB + \frac{\beta}{\alpha}$

$=$

$= 0$, seu $B = -\frac{\beta}{m\alpha}$; $mC + \frac{m(m-1)}{2}B^2 + (m+n)\frac{B\beta}{\alpha} + \frac{\gamma}{\alpha} = 0$, seu $C = \frac{(m+2n+1)\beta^2 - 2m\alpha\gamma}{2m^2\alpha^2}$, & ita porro. Quamobrem formula generalis quæsita erit

$$y = v^{\frac{1}{m}} - \frac{\beta}{m\alpha}v^{\frac{1+n}{m}} + \frac{(m+2n+1)\beta^2 - 2m\alpha\gamma}{2m^2\alpha^2}v^{\frac{1+2n}{m}} - \left(\frac{(2m^2+9mn+9n^2+3m+6n+1)\beta^3}{6m^3\alpha^3} + \frac{(m+3n+1)\beta\gamma}{m^2\alpha^2} - \frac{\delta}{m\alpha}\right)v^{\frac{1+3n}{m}} \text{ \&c.}$$

671. *Corell.* 1. Si fuerit $m = 1$, $n = 2$, seu $x = \alpha y + \beta y^3 + \gamma y^5 + \delta y^7 +$ &c. erit $y = v - \frac{\beta}{\alpha}v^3 + \frac{3\beta^2 - \alpha\gamma}{\alpha^2}v^5 + \frac{8\alpha\beta\gamma - \alpha^2\delta - 12\beta^3}{\alpha^3}v^7 -$ &c. seu

$$y = \frac{1}{\alpha}x - \frac{\beta}{\alpha^4}x^3 + \frac{3\beta^2 - \alpha\gamma}{\alpha^7}x^5 + \frac{8\alpha\beta\gamma - \alpha^2\delta - 12\beta^3}{\alpha^{10}}x^7 - \text{ \&c.}$$

672.

672. *Coroll.* 2. Si fuerit $x = y^{-\frac{1}{2}} - \frac{1}{2} y^{\frac{1}{2}} - \frac{1}{8} y^{\frac{3}{2}} - \frac{13}{16} y^{\frac{5}{2}}$ — &c. erit $\alpha = 1$; $\beta = -\frac{1}{2}$; $\gamma = -\frac{1}{8}$; $\delta = -\frac{13}{16}$; $v = x$, $m = -\frac{1}{2}$; $n = 1$; quapropter prodibit $y = \frac{1}{x^2} - \frac{1}{x^4} - \frac{1}{x^6} - \frac{1}{x^8}$ — &c.

Hoc problema *methodus serierum inversa* nuncupatur.

Exemplum VII. Data tangentis t expressione jam cognita

$$t = \frac{v - \frac{v^3}{2 \cdot 3} + \frac{v^5}{2 \cdot 3 \cdot 4 \cdot 5} - \&c.}{1 - \frac{v^2}{2} + \frac{v^4}{2 \cdot 3 \cdot 4} - \&c.}$$

invenire valorem v in terminis potestatum ipsius t.

Facta per denominatorem multiplicatione, terminisque transpositis, habetur $t = v + \frac{v^2 t}{2} - \frac{v^3}{2 \cdot 3} - \frac{v^4 t}{2 \cdot 3 \cdot 4} + \frac{v^5}{2 \cdot 3 \cdot 4 \cdot 5} + \&c.$

Pone $v = At + Bt^3 + Ct^5 +$ &c. erit
$v^2 = A^2t^2 + 2ABt^4 +$ &c.
$v^3 = A^3t^3 + 3A^2Bt^5 +$ &c.
$v^4 = A^4t^4 +$ &c.
$v^5 = A^5t^5 +$ &c. Hinc

$$t = \left\{ \begin{array}{lllll} v & = At & + Bt^3 & + Ct^5 & + \&c. \\ + \frac{v^2t}{2} & = \ldots\ldots & + \frac{A^2t^3}{2} & + ABt^5 & + \&c. \\ - \frac{v^3}{6} & = \ldots\ldots & - \frac{A^3t^3}{6} & - \frac{A^2Bt^5}{2} & - \&c. \\ - \frac{v^4t}{24} & = \ldots\ldots\ldots\ldots & & - \frac{A^4t^5}{24} & - \&c. \\ + \frac{v^5}{120} & = \ldots\ldots\ldots\ldots & & + \frac{A^5t^5}{120} & + \&c. \\ \&c. & & & \&c. & \end{array} \right.$$

Confer terminos homologos; ſtabit $A = 1$; $B = -\frac{1}{3}$; $C = \frac{1}{5}$; &c. ac proinde $v = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 -$ &c. Ergo ſi $v = arc.\ 45^\circ$; erit $t = 1$; adeoque $arc.\ 45^\circ = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} +$ &c. nempe formula

mula rursus Gregoriana, vel Leibnitiana, tam dicti arcus longitudinem, quam circuli quadraturam exprimens.

Exemplum VIII. Sit $\sqrt{1+x}$. Ergo $\sqrt{1+x} = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \&c.$ & quadrando

$$\begin{aligned} 1 + x = 1 + 2Ax &+ 2Bx^2 + 2ABx^3 + 2ACx^4 + \\ &+ A^2x^2 + 2Cx^3 \quad + B^2x^4 \quad + \&c. \\ &\qquad\qquad\qquad + 2Dx^4 + \end{aligned}$$

Hinc $A = \frac{1}{2}$; $B = -\frac{1}{8}$; $C = \frac{1}{16}$; $D = -\frac{5}{128}$; &c.

quare $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \&c.$

quæ series etiam provenit, si in formula $\sqrt{1+x}$ loco x collocetur $2x + xx$, ut fiat $\sqrt{1+2x+xx} = 1 + x =$

$$\begin{aligned} 1 + 2Ax + \; & Ax^2 + 4Bx^3 + \; Bx^4 + \; 6Cx^5 + \\ & + 4Bx^2 + 8Cx^3 + 12Cx^4 + 32Dx^5 + \; \&c. \\ & \qquad\qquad + 16Dx^4 + 32Ex^5 + \end{aligned}$$

ubi rursus $A = \frac{1}{2}$; $B = -\frac{1}{8}$; $C = \frac{1}{16}$; $D = -\frac{5}{128}$ + &c.

Aliter, ac generatim per methodum paulò ante adhibitam.

Pone $(1+x)^{\frac{1}{n}} = 1 + Ax + Bx^2 + Cx^3 + \&c.$

erit $\frac{1}{n} l(1+x) = l(1 + Ax + Bx^2 + Cx^3 + \&c.)$,

& reducendo in series, fiet

X 2 $\frac{x}{n}$

$$\frac{x}{n}-\frac{x^2}{2n}+\frac{x^3}{3n}-\&c.=\begin{array}{l}Ax+Bx^2+Cx^3+\\ \qquad -\frac{A^2x^2}{2}-ABx^3+\&c.\\ \qquad\qquad +A^3x^3+\end{array}$$

comparatisque terminis homologis, habebis $A=\frac{1}{n}$; $B=\frac{1-n}{2nn}$; $C=\frac{1+2nn-3n}{6n^3}$; &c. Unde

$$(1+x)^{\frac{1}{n}}=1+\frac{x}{n}+\frac{(1-n)}{2nn}x^2+\frac{(1+2nn-3n)}{6n^3}x^3$$

+ &c. Id autem pro casibus consimilibus semel præmonuisse sat esto.

673. *Coroll.* Finge $x=\frac{bx}{aa}$. Erit $\sqrt{a^2\pm bx}=$

$a\sqrt{1\pm\frac{bx}{aa}}=a\pm\frac{bx}{2a}-\frac{b^2x^2}{8a^3}\pm\frac{b^3x^3}{16a^5}-$ &c. prorsus ut alibi

Exemplum IX. Sit $\sqrt{1+xx}$. Ergo $\sqrt{1+xx}=1+Ax^2+Bx^4+Cx^6+$ &c. substituendo xx loco x in Exempli antecedentis prima æquatione. Quare quadrando fit

$$\begin{array}{l}1+xx=1+2Ax^2+A^2x^4+2Cx^6+\&c.\\ \qquad\qquad\qquad\quad +2Bx^4+2ABx^6+\&c.\end{array}$$

Hinc $A=\frac{1}{2}$; $B=-\frac{1}{2.4}$; $C=\frac{1.3}{2.4.6}$; &c. adeoque $\sqrt{1}$

$$\sqrt{1+xx} = 1 + \frac{1}{2}x^2 - \frac{1 \cdot 1}{2 \cdot 4}x^4 + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}x^6 - \&c.$$

Ergo statuendo $xx = \pm \frac{nxx}{aa}$, provenit $a\sqrt{1 \pm \frac{nxx}{aa}}$

$$= \sqrt{aa \pm nxx} = a \pm \frac{nx^2}{2a} - \frac{n^2x^4}{8a^3} \pm \frac{n^3x^6}{16a^5} - \&c.$$

674. *Scholion.* Ex hoc Exemplo IX. sequens haud inelegans demonstratur

Theorema. Sint duo solida similia, & inæqualia. Sint præterea duo solida ipsis similia, & invicem æqualia, ita ut horum superficies simul sumptæ solidorum inæqualium superficies simul sumptas adæquent. Si solidorum æqualium dimidia superficies statuatur $= 1$, & solidorum inæqualium altera superficies ponatur $= 1 + xx$, erit altera $= 1 - xx$. Hisce stantibus, ajo, solidorum æqualium aggregatum esse *minimum.*

Quum enim solidorum æqualium soliditas sit ad inæqualium soliditatem, ut 2 ad $(1+xx)\sqrt{1+xx} + (1-xx)\sqrt{1-xx}$, sufficit ostendere, esse $2 < (1+xx)\sqrt{1+xx} + (1-xx)\sqrt{1-xx}$. Pro quo radicales $\sqrt{1+xx}$, $\sqrt{1-xx}$ resolvantur in series; tum instituta multiplicatione per $1+xx$ cum prima, & per $1-xx$ cum secunda serie, fiat summa; prodibit $2 < 2 + \frac{x^4}{4} + \frac{6x^8}{128} + \&c.$ *Quod erat* &c.

Idem per elementarem quoque Geometriam præstari breviter potest, & ad quemlibet solidorum numerum deinde traduci.

Exemplum X. Sit $\frac{1}{\sqrt{1+xx}}$. Ergo

$\frac{1}{\sqrt{1+xx}} = 1 + Ax^2 + Bx^4 + Cx^6 +$ &c. Quapropter

$$\frac{1}{1+xx} = 1 + 2Ax^2 + A^2x^4 + 2Cx^6 + \&c. + 2Bx^4 + 2ABx^6 + \&c.$$

& multiplicando per $1 + x^2$, habetur

$$1 = 1 + 2Ax^2 + A^2x^4 + 2Cx^6 + x^2 + 2Bx^4 + 2ABx^6 + \&c. + 2Ax^4 + A^2x^6 + 2Bx^6$$

ex quo colligitur $A = -\frac{1}{2}$; $B = \frac{3}{8}$; $C = -\frac{15}{48} +$ &c. adeoque

$$\frac{1}{\sqrt{1+xx}} = 1 - \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \&c.$$

Exemplum XI. Sit extrahenda radix quadrata ex formula
$a^2 + 4abx + (6ac + 4b^2)x^2 + 12bcx^3 + 9c^2x^4$. Fiat igitur

$\sqrt{a^2 + 4abx + (6ac + 4b^2)x^2 + 12bcx^3 + 9c^2x^4} = A + Bx + Cx^2 + Dx^3 + Ex^4 +$ &c. & quadrando, stabit

a^2

$$a^2 + 4abx + (6ac + 4b^2)x^2 + 12bcx^3 + 9c^2x^4 =$$
$$A^2 + 2ABx + 2ACx^2 + 2ADx^3 + 2AEx^4$$
$$+ B^2x^2 + 2BCx^3 + 2BDx^4 + \&c.$$
$$+ C^2x^4$$

factaque de more terminorum comparatione, invenietur $A = a$; $2AB = 4ab$, ergo $B = 2b$; $2AC + B^2 = 6ac + 4bb$; ergo $C = 3c$; $2AD + 2BC = 12bc$, ergo $D = 0$; itidemque $E = 0$ &c. Radix igitur quæsita $= a + 2bx + 3cxx$, ut examine instituto comprobatur.

Exemplum XII. Sit $\sqrt[3]{a^3 + 3a^2b + 3ab^2 + b^3}$, quæ statuatur $= A + Bb + Cb^2 + Db^3 +$ &c. Fiat cubus, eritque

$$a^3 + 3a^2b + 3ab^2 + b^3 = A^3 + 3A^2Bb + 3A^2Cb^2 + 3A^2Db^3$$
$$+ 3AB^2b^2 + 6ABCb^3 + \&c.$$
$$+ B^3b^3$$

Per consuetam comparationem nanciscimur $A = a$, $B = 1$, $C = 0$, $D = 0$ &c. ergo radix cubica expetita $= a + b$.

675. *Corollarium.* Ex hoc, & præcedente Exemplo liquet, quod si dentur quantitates generis expositi, quæ sint potestates exactæ cujuscumque ordinis, earum radices facile per hanc methodum poterunt obtineri.

676. Propositio II. *Fractionem genuinam in fractiones partiales dispertiri.*

Resolutio I. Fractio genuina, quæ sic exprimi potest, $\frac{X}{x(a+x)(b+x)(c+x)\&c.}$ ponatur $= \frac{A}{x} +$

$\frac{B}{a+x} + \frac{C}{b+x} + \frac{D}{c+x} +$ &c. deinde facta ad idem nomen reductione, (seu multiplicando $\frac{A}{x} + \frac{B}{a+x} + \frac{C}{b+x} + \frac{D}{c+x} +$ &c. per $x(a+x)(b+x)(c+x)$ &c. ita ut prodeat $X = A(a+x)(b+x)(c+x)$ &c. $+ Bx(b+x)(c+x)$ &c. $+ Cx(a+x)(c+x)$ &c. $+ Dx(a+x)(b+x)$ &c. $+$ &c.) comparentur termini, ut in superioribus Exemplis egimus, & indeterminatæ A, B, C, D, &c. (fractionum nempe partialium numeratores), dummodo $x, a+x, b+x, c+x$, &c. sint numeri inter se primi, definientur.

II. Si fractionis propositæ denominator pluribus constet factoribus invicem æqualibus, denominator iste considerari debet perinde ac si unum tantum contineat ex factoribus invicem æqualibus, deinde rescissa fractione proposita in fractiones partiales, per denominatores omissos multiplicentur fractionum partialium denominatores. Tunc autem si quis ex hujusmodi factoribus, potestatibus exceptis, sit numerus compositus, per expositam methodum denuo institutam data fractio tandem in fractiones partiales simplices rescindetur. Sed clarius per Exempla.

Exemplum I. Esto $\frac{2x^2 - 6x - 2}{x(x-1)(x+2)}$. Ergo

2x

$\frac{2x - 6x - 2}{x(x-1)(x+2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$; adeoque

$2x^2 - 6x - 2 = (x^2 + x - 2)A + (x^2 + 2x)B + (x^2 - x)C = -2A + \begin{matrix} Ax + Ax^2 \\ + 2Bx + Bx^2 \\ - Cx + Cx^2 \end{matrix}$

Hinc $-2A = -2$, seu $A = 1$; $2B = -6 - A + C = -7 + C$; $C = 2 - A - B = 1 - B$; ergo $B = -2$; $C = 3$. Itaque $\frac{2x^2 - 6x - 2}{2(x-1)(x+2)} = \frac{1}{x} - \frac{2}{x-1} + \frac{3}{x+2}$.

Exemplum II. Esto $\frac{2x}{(1+x)^2(1-x)}$. Facta ad eundem denominatore reductione, post æquationem $\frac{2x}{(1+x)(1-x)} = \frac{A}{1+x} + \frac{B}{1-x}$, habetur $2x = \begin{matrix} A - Ax; \\ + B + Bx \end{matrix}$ unde $A + B = 0$, seu $A = -B$; $-A + B = 2$, seu $B = 1 = A$; ergo $\frac{2x}{(1+x)(1-x)} = \frac{1}{1+x} + \frac{1}{1-x}$; quare denominatoribus istis per omissum factorem $1 + x$ multiplicatis, stabit $\frac{2x}{(1+x)^2(1-x)} = \frac{1}{(1+x)^2} + \frac{1}{(1+x)(1-x)}$. Restat dispertienda fra-

fractio $\frac{1}{(1+x)(1-x)}$; quapropter statue

$$\frac{1}{(1+x)(1-x)} = \frac{A}{1+x} + \frac{B}{1-x},$$ sive ad idem nomen reducendo, $1 = A - Ax + B + Bx$; hinc $A + B = 1$, & $A = B$; ac proinde $A = \frac{1}{2}$, $B = \frac{1}{2}$; ergo

$$\frac{1}{(1+x)(1-x)} = \frac{\frac{1}{2}}{1+x} + \frac{\frac{1}{2}}{1-x};$$ adeoque

$$\frac{2x}{(1+x)^2(1-x)} = \frac{1}{(1+x)^2} + \frac{\frac{1}{2}}{1+x} + \frac{\frac{1}{2}}{1-x}.$$

Exemplum III. Esto $\frac{1}{tv}$. Pone $t = x + a$, $v = x + b$. Erit $\frac{1}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$; & agendo ut supra, habetur $1 = Ab + Ax + Ba + Bx$. Hinc $Ab + Ba = 1$; $A = -B$; ex quo elicitur $A = \frac{1}{b-a}$; $B = \frac{1}{a-b}$. Ergo $\frac{1}{(x+a)(x+b)} =$

$\frac{1}{(x+a)(b-a)} + \frac{1}{(x+b)(a-b)}$. Sed $b - a = v - t$,

$-t$, & $a-b=t-v$. Ergo $\frac{1}{tv}=\frac{1}{t(v-t)}+\frac{1}{v(t-v)}$.

677. *Coroll.* Hinc fractionis, cujus numerator unitas, denominator productum ex quovis factorum numero, resciſſio in fractiones partiales. Esto enim primum fractio $\frac{1}{tvy}$ scindenda, ut dictum. Ob $\frac{1}{tvy}=\frac{1}{tv}\times\frac{1}{y}$, erit $\frac{1}{tvy}=\frac{1}{ty(v-t)}+\frac{1}{vy(t-v)}$.

Sed $\frac{1}{ty}\times\frac{1}{v-t}=\frac{1}{t(y-t)(v-t)}+\frac{1}{y(t-y)(v-t)}$, pariterque $\frac{1}{vy}\times\frac{1}{t-v}=\frac{1}{v(y-v)(t-v)}+\frac{1}{y(v-y)(t-v)}$; ergo $\frac{1}{tvy}$

$$=\frac{1}{t(y-t)(v-t)}+\frac{1}{y(t-y)(v-t)}+\frac{1}{v(y-v)(t-v)}+\frac{1}{y(v-y)(t-v)}.$$ Est autem

$$\frac{1}{y(t-y)(v-y)}=\frac{1}{y(t-y)(v-t)}+$$

$\frac{1}{y(v-y)(t-v)}$, ut conſtat ponendo $t-y$ pro t, & $v-y$ pro v, nec non multiplicando per $\frac{1}{y}$. Ergo $\frac{1}{tvy} = \frac{1}{t(v-t)(y-t)} + \frac{1}{v(t-v)(y-v)} + \frac{1}{y(t-y)(v-y)}$.

Eſto deinde fractio $\frac{1}{tvyz}$. Per demonſtrata habetur $\frac{1}{tvy} \times \frac{1}{z} = \frac{1}{tz(v-t)(y-t)} + \frac{1}{vz(t-v)(y-v)} + \frac{1}{yz(t-y)(v-y)}$. Itaque reſolutis $\frac{1}{tz}$, $\frac{1}{vz}$, $\frac{1}{yz}$ per exemplum præſens in fractiones partiales, fit

$$(A) \ldots\ldots \frac{1}{tz(v-t)(y-t)} = \frac{1}{t(z-t)(v-t)(t-y)} + \frac{1}{z(t-z)(v-t)(y-t)}$$

(B)

$$(B) \ldots\ldots \frac{1}{vz(t-v)(y-v)} = \frac{1}{v(z-v)(t-v)(y-v)}$$

$$+ \frac{1}{z(v-z)(t-v)(y-v)}$$

$$(C) \ldots\ldots \frac{1}{yz(t-y)(v-y)} = \frac{1}{y(z-y)(t-y)(v-y)}$$

$$+ \frac{1}{z(y-z)(t-y)(v-y)};$$

Ergo $\frac{1}{tvyz}$ adæquat tria ſecunda membra æquationum (A), (B), (C); Sed poſitis $t-z$ pro t, $v-z$ pro v, $y-z$ pro y, & multiplicando poſtmodum per $\frac{1}{z}$, prodit [*per dem.*] $\frac{1}{tvy} \times \frac{1}{z} =$

$$\frac{1}{z(t-z)(v-z)(y-z)} = \frac{1}{z(t-z)(v-t)(y-t)}$$

$$+ \frac{1}{z(v-z)(t-v)(y-v)} + \frac{1}{z(y-z)(t-y(v-y)};$$

Ergo $\frac{1}{tvyz} = \frac{1}{t(v-t)(y-t)(z-t)} +$

1

$$\frac{1}{v(t-v)(y-v)(z-v)}+\frac{1}{y(t-y)(v-y)(z-y)}$$

$+\frac{1}{z(t-z)(v-z)(y-z)}$; atque ita porro.

Fractiones autem partiales inventas fractionibus propositis $\frac{1}{tv}$, $\frac{1}{tvy}$, $\frac{1}{tvyz}$ respondentes vel leviter perpendenti facillima statim occurret pro quibuslibet hujusmodi fractionibus partialibus reperiundis

Regula Generalis. (1.°) Multiplica primum denominatoris factorem per reliquorum cum primo differentias invicem ductas; hic erit primæ fractionis denominator, cujus numerator unitas.

(2.°) Multiplica secundum denominatoris factorem per reliquorum cum secundo differentias invicem ductas; hic erit secundæ fractionis denominator, cujus numerator unitas.

(3.°) Multiplica tertium denominatoris factorem per reliquorum cum tertio differentias invicem ductas; hic erit tertiæ fractionis denominator, cujus numerator unitas.

(4.°) Et sic deinceps, donec factorum in fractionis propositæ denominatore existentium numerus exhauriatur.

Vel Brevius. Quæcumque ex fractionibus partialibus pro denominatore unum habebit ex datis factoribus ductum in omnia binomia, quæ prodeunt ipsum a reliquis singillatim subtrahendo.

678.

678. *Scholion.* (1.°) Si fractionis denominator pluribus constaret factoribus invicem æqualibus, veluti $\frac{1}{xyy}$, prodiret $\frac{1}{xyy} = \frac{1}{x(y-x)(y-x)} +$

$$\frac{1}{y(x-y)(y-y)} + \frac{1}{y(x-y)(y-y)} =$$

$\frac{1}{x(y-x)^2}$, quod foret absurdum. Itaque ut id incommodi vitetur, denominator considerari debet æque ac si unum tantum contineat ex factoribus invicem æqualibus; tunc autem rescissa prius fractione, per denominatores postea prætermissos multiplicentur, ut nuper monuimus, fractionum partialium denominatores. Sic omissa y fractio $\frac{1}{xyy}$ evadit $\frac{1}{xy} =$

$$\frac{1}{x(y-x)} + \frac{1}{y(x-y)}, \text{ adeoque } \frac{1}{xyy} =$$

$$\frac{1}{xy(y-x)} + \frac{1}{y^2(x-y)}.$$

(2.°) Si omnes denominatoris factores ad eandem potestatem sint elati, eadem hic quoque locum habebit operatio. Sic fractio $\frac{1}{(x+a)^n\,(y+b)^n}$ resolvitur

in

in fractiones $\frac{1}{(x+a)^n \times \overline{(y+b)^n - (x+a)^n}}$ +

$\frac{1}{(y+b)^n \times \overline{(x+a)^n - (y+b)^n}}$, quod verum reperietur,

si fiat $[x+a]^n = t$, $[y+b]^n = v$.

(3.°) Æquatio $\frac{1}{tv} = \frac{1}{t[v-t]} + \frac{1}{v[t-v]}$ etiam sic demonstratur. Mutentur signa supra & infra in alterutra fractione secundi membri, ita ut fiat $\frac{1}{t[v-t]} - \frac{1}{v[v-t]}$, tum ad eundem denominatorem reducendo, proveniet $\frac{[v-t]^2}{tv[v-t]^2}$ $= \frac{1}{tv}$.

(4.°) Si fractionis numerator non sit unitas, sed quantitas aliqua in unitatem ducta; hac quantitate omissa, factaque in fractiones partiales resolutione, singulis istis factoribus dicta quantitas omissa restituatur, elisis postea per divisionem elidendis.

(5.°) Cum fractio aliqua puta $\frac{1}{[x+a][x-b][x+c]}$ scinditur in fractiones partiales, vitantur series recurrentes altiorum ordinum, quæ per propositionis præcedentis methodum occurrerent. (6.°)

(6.°) Hae scindendarum fractionum methodi hoc præstant, quod si dicti denominatoris factores una efformentur variabili, & diversis constantibus, puta $x+a$, $x+b$, $x+c$, &c. singulæ fractiones partiales ipsam variabilem ad primam tantummodo potentiam assurgentem in denominatore retinebunt. Sic

$$\frac{1}{(x-a)(x-b)(x-c)} = \frac{1}{(x-a)(a-b)(a-c)}$$

$$+\frac{1}{(x-b)(b-a)(b-c)} + \frac{1}{(x-c)(c-a)(c-b)},$$

quod in Calculo Integrali egregiam adfert utilitatem.

(7.°) Eximiam summus Eulerus exhibet methodum in Cap. II. suæ *Introduct. in analysin infinitorum*, cujus Tiro compos fieri debet, cujusque en fundamentum.

Sit fractio $\frac{P}{(p-qz)^n}$, *in qua denominatoris factores sunt invicem æquales, & maxima incognitæ* z *potestas in numeratore sit minor maxima incognitæ potestate in denominatore; invenire fractiones partiales, ex quarum additione fractio proposita coalescit.*

Pro quo quum maxima incognitæ z. potestas in P esse debeat (ut fractio sit genuina) minor quam z^n, numerator repræsentari poterit per

$$\alpha + \beta z + \gamma z^2 + \delta z^3 + \ldots\ldots + \omega z^{n-1}.$$

Jam vero quantitas ista reduci semper poterit ad formam

$A + B[p - qz] + C[p - qz]^2 + D[p - qz]^3 + \dots$

$+ K[p - qz]^{n-1}$, nam evolvendo poteſtates, ac deinde ordinando, emergit quantitas ejuſdem formæ.

Ergo erit $\frac{P}{[p - qz]^n} =$

$$\frac{A + B(p - qz) + C(p - qz)^2 + D(p - qz)^3 + \dots + K(p - qz)^{n-1}}{(p - qz)^n}$$

$$= \frac{A}{[p - qz]^n} + \frac{B}{[p - qz]^{n-1}} + \frac{C}{[p - qz]^{n-2}}$$

$+ \frac{D}{[p - qz]^{n-3}} + \dots + \frac{K}{p - qz}$, quæ ſunt fractiones partiales quæſitæ.

679. Propoſitio III. *Idem Propoſitionis antecedentis Problema brevius expedire.*

Reſolutio. Poſt indicatam ad idem nomen reductionem, ſeu factorum in denominatore datæ fractionis exiſtentium multiplicationem, ſinguli iſti factores ponantur æquales zero, ac ſucceſſive exploretur per ipſos æquatio, quæ indeterminatarum A, B, C, &c. valores ſingillatim exhibebit.

Exemplum. Proponatur rurſus fractio $\frac{2x^2 - 6x - 2}{x[x-1][x+2]}$, ac deventum jam ſit, ut diximus, ad æquationem $2x^2 - 6x - 2 = [x^2 + x - 2]A + [x^2 + 2x]B + [x^2 - x]C$. Denominatoris factores x, $x - 1$, $x + 2$ ponantur ſucceſſive æquales zero, & ſinguli in æquatio-

tionem allatam introducantur. Jam vero si statuatur $x=0$, erunt $[x^2+2x]B=0$, $[x^2-x]C=0$, quare æquatio evadet $-2=-2A$, seu $A=1$. Si fiat $x-1=0$, erit $x=1$, adeoque $[x^2+x-2]A=0$, $[x^2-x]C=0$, & $2-6-2=-6=[1+2]B$, seu $B=-2$. Si ponatur $x+2=0$, erit $x=-2$, hinc $[x^2+x-2]A=0$ $[x^2+2x]B=0$, ac proinde $8+12-2=18=[4+2]C$, seu $C=3$; prorsus ut supra; & sic de ceteris.

680. *Coroll.* Si factores in fractionis datæ denominatore existentes sint trinomia ex radicibus imaginariis conflata, finge, esse

$$\frac{x^m}{[a+bx+cx^2][e+fx+gx^2][h+kx+lx^2]\&c.}$$

$$=\frac{A+Bx}{a+bx+cx^2}+\frac{C+Dx}{e+fx+gx^2}+\frac{E+Fx}{h+kx+lx^2}$$

+ &c. dummodo m sit semper minor quam incognitæ x exponens ortus ex trinomiorum multiplicatione, ac pone $[e+fx+gx^2][h+kx+lx^2]=Q$; $[a+bx+cx^2][h+kx+lx^2]=R$; $[a+bx+cx^2][e+fx+gx^2]=S$. Vi methodi generalis erit $x^m=[A+Bx]Q+[C+Dx]R+[E+Fx]S$; Igitur si fiat $a+bx+cx^2=0$, erunt $R=0$, $S=0$, ac proinde $x^m=[A+Bx]Q$. Statue nunc, æquationis $a+bx+cx^2=0$ factores imaginarios esse $t+rx=0$, $p+qx=0$; erit $x=-\frac{t}{r}$, & $x=-\frac{p}{q}$.

Hinc sumpta $x = -\frac{t}{r}$, erit $Q = (e + f \times -\frac{t}{r} + g \times \frac{t^2}{r^2})(b + k \times -\frac{t}{r} + l \times \frac{t^2}{r^2})$; quod si accipiatur altera radix $x = \frac{-p}{q}$ alius ipsius Q obtinebitur valor, qui dicatur Q'.

Substituantur isti valores in æquatione $x^m = (A + Bx)Q$, ac duæ sequentes emergent æquationes

$$\frac{-t^m}{r^m} = (A + Bx)Q \qquad A = \frac{-t^m}{r^m Q} + \frac{t}{r}B$$

adeoque

$$\frac{-p^m}{q^m} = (A + Bx)Q' \qquad A = \frac{-p^m}{q^m Q'} + \frac{p}{q}B$$

Habemus ergo $\frac{-t^m}{r^m Q} + \frac{t}{r}B = \frac{-p^m}{q^m Q'} + \frac{p}{q}B$, unde ipsius B valor elici potest, quo valore in æquationibus ad A subrogato, ejusdem A valor prodibit. Itaque si consimili modo alia trinomia tractentur, aliarum indeterminatarum C, D, &c. valores invenientur.

681. Propositio IV. *Æquationum radices per approximationem investigare.*

Resolutio. I. Statue incognitæ valorem æqualem seriei indeterminatæ $A + Bn + Cn^2 + Dn^3 + En^4 +$ &c. ita ut n exprimat quantitatem cognitam præ reliquis cognitis æquationem ingredientibus minimam.

Dein-

Deinde elevata hac ſerie ad gradus omnes, quos habet incognita in æquatione data, ſinguli novæ hinc ortæ æquationis termini, ut æquatio fiat identica, ponatur $=0$; ipſarum indeterminatarum A, B, C, D, E, &c. valores ſic elicientur; quibus de more ſubſtitutis, incognita, ſeu radix quæſita per ſeriem convergentem determinabitur.

II. Quod ſi duæ in æquatione data occurrant incognitæ, quod ſæpe contingit, tunc ſeries efformetur ex incognita, quæ ſit altera minor, ut nova emergat ſeries convergens; ex. gr. ſi fuerit $y < x$, adſumatur ſeries $x = A + By + Cy^2 + Dy^3 + Ey^4 + \&c.$ ac reliqua ut ſupra peragantur. *Quod erat* &c.

Exemplum I. Eſto $x^3 - qx \mp r = 0$, ſitque $r < q$. Finge $x = A + Br + Cr^2 + Dr^3 + Er^4 + \&c.$ Erit

$$\left.\begin{array}{r|l} x^3 & = A^3 + 3A^2Br + 3A^2Cr^2 + 3A^2Dr^3 + 3A^2Er^4 \\ & \qquad + 3AB^2r^2 + 6ABCr^3 + 6ABDr^4 \\ & \qquad\qquad + B^3r^3 + 3AC^2r^4 \\ & \qquad\qquad\qquad + 3B^2Cr^4 \\ -qx & = -Aq - Bqr - Cqr^2 - Dqr^3 - Eqr^4 \\ \mp r & = \mp r \end{array}\right\} \&c. = 0$$

Singulis terminis poſitis $=0$, ſtabit

(1.°) $A^3 - Aq = 0$; $A^2 = q$; unde $A = \pm\sqrt{q}$, & ſigno poſitivo retento,

(2.°) $3A^2B - Bq \mp 1 = 0$; $B = \pm \frac{1}{3A^2 - q} = \pm \frac{1}{2q}$.

(3.°) $3A^2C + 3AB^2 - Cq = 0$; $C = -\frac{3AB^2}{3A^2 - q} = \frac{-3}{8q^2\sqrt{q}}$.

(4.°) $3A^2D + 6ABC + B^3 - Dq = 0$; $D = -\frac{(6ABC + B^3)}{3A^2 - q} = \left(\frac{-9}{8q^3} + \frac{1}{8q^3}\right) : 2q = \pm \frac{1}{2q^4}$.

(5.°) $3A^4E + 6ABD + 3AC^2 + 3B^2C - Eq = 0$; $E = -\frac{(6ABD + 3AC^2 + 3B^2C)}{3A^2 - q} = -\frac{105}{128q^5\sqrt{q}}$; &c.

quapropter $x = \sqrt{q} \pm \frac{r}{2q} - \frac{3r^2}{8q^2\sqrt{q}} \pm \frac{r^3}{2q} - \frac{105}{128q^5\sqrt{q}} \pm$ &c.

682. *Scholion.* Si radix inventa expressa per decimales ponatur $= \pm \omega$, erit $x = \pm \omega$; quare dividendo per hujusmodi quantitates æquationem datam, prodibit $x^2 = q \pm \frac{r}{\pm \omega}$, & $x = \pm \sqrt{q \pm \frac{r}{\pm \omega}}$. Hinc in casu quoque irreductibili, ubi radices sunt omnes reales inæquales, & irrationales, istæ poterunt

runt per approximationem sine sinuum vel cosinuum ope inveniri, nam $\sqrt{q - \frac{r}{\omega}} = \sqrt{\frac{\alpha q - r}{\omega}}$ est semper quantitas realis ob $q > r$.

Exemplum II. Esto $x^3 + bbx - 2b^3 = 0$, sitque a
$+ abx - a^3$
omnium cognitarum minima; Ergo $x = A + Ba + Ca^2 - Da^3 + Ea^4 + \&c.$ Hinc

$$\left.\begin{array}{r|l}
x^3 = & A^3 + 3A^2Ba + 3AB^2a^2 + B^3a^3 \quad + 3AC^2a^4 \\
 & \qquad + 3A^2Ca^2 + 3A^2Da^3 + 3B^2Ca^4 \\
 & \qquad\qquad + 6ABCa^3 + 3A^2Ea^4 \\
 & \qquad\qquad\qquad + 6ABDa^4 \\
+ bbx = & + Abb + Bb^2a + Cb^2a^2 + Db^2a^3 + Eb^2a^4 \\
+ abx = & \quad + Aba + Bba^2 + Cba^3 + Dba^4 \\
- 2b^3 = & - 2b^3 \\
- a^3 = & \qquad - a^3
\end{array}\right\} \&c. = 0$$

singulisque terminis positis $= 0$, prodibit

(1.°) $A^3 + Abb - 2b^3 = 0$, ubi quum divisio per $A - b$ fiat exacta, habetur $A = b$.

(2.°) $3A^2B + Bb^2 + Ab = 0$, unde $B = \frac{-Ab}{3A^2 + b^2} = -\frac{1}{4}$.

 (3.°)

(3.°) $3AB^2 + 3A^2C + Cb^2 + Bb = 0$, & $C = \frac{-Bb - 3AB^2}{3A^2 + bb} = \frac{1}{64b}$.

(4.°) $B^3 + 3A^2D + 6ABC + Db^2 + Cb - 1 = 0$, & $D = \frac{-Cb - B^3 - 6ABC + 1}{3A^2 + bb} = \frac{131}{512b^2}$.

(5.°) $3AC^2 + 3B^2C + 3A^2E + 6ABD + Eb^2 + DB = 0$, quare $E = \frac{-DB - 6ABD - 3B^2C - 3AC^2}{3A^2 + bb} = \frac{509}{16384b^3}$. &c.

factaque imperata substitutione provenit

$$x = b - \frac{1}{4}a + \frac{1}{64}\cdot\frac{a^2}{b} + \frac{131}{512}\cdot\frac{a^3}{b^2} + \frac{509}{16384}\cdot\frac{a^4}{b^3} \text{ \&c.}$$

683. Propositio V. *Ex aequatione in infinitum excurrente radicem extrahere.*

Resolutio. Esto $z = tx + sx^2 + rx^3 + qx^4 + px^5 +$ &c. Fiat $x = Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 +$ &c. erit

$$\left.\begin{array}{l|lllll}
x^2 = & A^2z^2 & + 2ABz^3 & + B^2z^4 & + 2BCz^5 & \\
 & & & + 2AC \ldots & + 2AD \ldots & \\
x^3 = & & + A^3 \ldots\ldots & + 3A^2B \ldots & + 3AB^2 \ldots & \\
 & & & & + 3A^2C \ldots & \\
x^4 = & & & + A^4 \ldots\ldots & + 4A^3B \ldots & \\
x^5 = & & & & + A^5 \ldots\ldots & \\
\text{\&c.} & & & \text{\&c.} & &
\end{array}\right\} \text{\&c.}$$

Suffectis hisce valoribus in æquatione $-z+tx+sx^2+rx^3+qx^4+px^5+$ &c. prodibit

$$\left.\begin{array}{r|lllll}
-z= & -z & & & & \\
+tx= & Atz+Btz^2 & +Ctz^3 & +Dtz^4 & +Etz^5 & \\
+sx^2= & & +A^2s\ldots & +2ABs\ldots & +B^2s\ldots & +2BCs\ldots \\
 & & & & +2ACs\ldots & +2ADs\ldots \\
+rx^3= & & & +A^3r\ldots & +3A^2Br\ldots & +3AB^2r\ldots \\
 & & & & & +3A^2Cr\ldots \\
+qx^4= & & & & +A^4q\ldots & +4A^3Bq\ldots \\
+px^5= & & & & & +A^5p\ldots \\
\text{\&c.} & \text{\&c.} & & & &
\end{array}\right\}\text{\&c.}$$

Fiant ut in præcedente singuli termini nihilo æquales, erit

(1.°) $At-1=0;\ A=\frac{1}{t}$.

(2.°) $Bt+A^2s=0;\ B=-\frac{s}{t^3}$.

(3.°) $Ct+2ABs+A^3r=0;\ C=\frac{2s^2-rt}{t^5}$.

(4.°) $Dt+B^2s+2ACs+3A^2Br+A^4q=0;\ D$ $=\frac{5rst-5s^3-t^2q}{t^7}$.

(5.°) $Et+2BCs+2ADs+3AB^2r+3A^2Cr$ $+4A^3Bq+A^5p=0;\ E=$

$$\frac{14s^4 - 21ts^2r + 6t^2sq + 3t^2r^2 - t^3p}{t^9} \text{ . \&c.}$$

Qui valores si in æquatione $x = Az + Bz^2 + Cz^3$ $+$ &c. substituantur, habetur radix quæsita

$$x = \frac{z}{t} - \frac{s}{t^3} z^2 + \frac{(2s^2 - rt)}{t^5} z^3 + \frac{(5rst - 5s^3 - t^2q)}{t^7} z^4$$

$$+ \frac{(14s^4 - 21ts^2r + 6t^2sq + 3t^2r^2 - t^3p)}{t^9} z^5$$

&c. *Quod erat* &c.

684. *Coroll.* Hinc dato logarithmo numeri $1 + x$, ipse numerus $1 + x$ obtinetur. Sit enim $z = l(1 + x)$, erit $z = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 +$ &c. (257.). Sed hic $t = 1$, $s = \frac{1}{2}$, $r = \frac{1}{3}$, $q = \frac{1}{4}$, &c. ergo in valore generali ipsius x substitutis valoribus, reperietur $x = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 +$ &c. $= \frac{z}{1} +$ $\frac{z^2}{1 \cdot 2} + \frac{z^3}{1 \cdot 2 \cdot 3} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4} +$ &c. adeoque $1 + x$ $= 1 + \frac{z}{1} + \frac{z^2}{1 \cdot 2} + \frac{z^3}{1 \cdot 2 \cdot 3} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4} +$ &c. Quod si dato logarithmo numeri $1 - z$, iste numerus

rus optetur, statuendo $z = l(1 - x)$, & operando ut supra, invenietur $1 - x = 1 - \frac{z}{1} + \frac{z^2}{1.2} - \frac{z^3}{1.2.3} + \frac{z^4}{1.2.3.4} -$ &c. Ex quo coniicitur, methodum istam perinde atque illam, quam methodum serierum inversam paulo ante vocavimus (672.), eodem collineare.

CAPUT III.

DE SERIEBUS ARITHMETICIS SUMMAM ALGEBRAICAM RECIPIENTIBUS, UBI DE NUMERIS FIGURATIS.

Definitiones.

685. Series Arithmeticæ, in quibus, sumptis successive duorum adhærentium terminorum differentiis, devenitur ad differentias constantes, vocantur etiam *Algebricæ*.

686. *Functio* alicujus quantitatis variabilis, seu indeterminatæ est expressio quocumque modo composita ex eadem indeterminata, & quantitatibus constantibus. Sic $a + 5n$; $2ab + 3an + nn$; $4ab + \sqrt{an - nn}$; a^n; &c. sunt functiones variabilis n.

687. Numerus indeterminatus omnium terminorum seriem aliquam, vel ejus portionem constantium, di-

dicatur n. Functio ipsius n vocatur seriei *terminus generalis*, si substitutis in eo pro n successive numeris naturalibus 1, 2, 3, 4, &c. singuli seriei termini separatim habentur. Sic seriei 1, 7, 13, 19, 25, 31, &c. terminus generalis est $6n-5$, quia si in locum n sufficias successive numeros naturales, singulos seriei terminos invenies.

688. *Terminus summatorius*, seu *summa generalis* seriei est functio ipsius n, in quo si pro n numerus integer positivus subrogetur, tot terminorum seriei (incipiendo a primo) summa obtinetur, quot numerus iste continet unitates. Sic seriei antecedentis 1, 7, 13, 19, &c. summa generalis est $3n^2-2n$, quia si numerorum, puta quinque, summam exoptes, pone $n=5$, eritque $3n^2-2n=65$.

689. Propositio I. *Series Arithmetica, vel Algebrica cujuscumque ordinis generatim repræsentari potest*

per $A+Bn+Cn^2+Dn^3 \ldots\ldots + Pn^m$.

Dem. Jam indicavimus (502.) quid per series arithmeticas primi, secundi, tertii &c. ordinis intelligi debeat. Itaque si fuerint series b, $2b$, $3b$, $4b$, &c.; bc, $2bc$, $3bc$, $4bc$, &c.; $a+bc$, $a+2bc$, $a+3bc$, $a+4bc$, &c. istæ, ut patet, erunt series arithmeticæ primi ordinis, in quibus primæ terminorum differentiæ sunt constantes, ac proinde quælibet ex istis seriebus, itidemque plurium hujusmodi serierum aggregatum designari poterit per $A+Bn$.

Si fuerit series composita ex quadratis b^2, $4b^2$, $9b^2$, $16b^2$, $25b^2$, &c. hæc erit series arithmetica secundi ordinis, habet enim secundas differentias constan-

stantes, quod primæ differentiæ seriem arithmeticam primi ordinis constituunt.

En exemplum b^2, $4b^2$, $9b^2$, $16b^2$, $25b^2$, &c.

$3b^2$, $5b^2$, $7b^2$, $9b^2$

$2b^2$, $2b^2$, $2b^2$

Similiter si fuerit series cb^2, $4cb^2$, $9cb^2$, $16cb^2$, $25cb^2$, &c. pariterque $a+cb^2$, $a+4cb^2$, $a+9cb^2$, $a+16cb^2$, $a+25cb^2$, &c. secundæ differentiæ semper evadent constantes, nempe $2cb^2$; quare series arithmetica secundi ordinis designabitur per $A+Cn^2$. Sed secundæ differentiæ constantes etiam reperientur, si fiat aggregatum seriei cujuslibet arithmeticæ primi ordinis cum serie qualibet arithmetica ordinis secundi; ergo series arithmetica secundi ordinis repræsentari quoque poterit per $A+Bn+Cn^2$. Eadem operatio iteretur quoad seriem arithmeticam tertii ordinis, & ista invenietur, exprimi posse per $A+Bn+Cn^2+Dn^3$; & sic deinceps. Ergo generatim series arithmetica cujuscumque ordinis repræsentabitur per $A+Bn+Cn^2+Dn^3 \dots + Pn^m$, ubi m denotat ordinem seriei, de qua est sermo. *Quod erat* &c.

690. Propositio II. *Series quælibet arithmetica, vel algebrica cujuscumque ordinis est series recurrens.*

Dem. Designet $A+B+C+D \dots\dots +Z$ seriem algebricam primi ordinis habentem $A+Bn$ pro termino generali. Per demonstrata in Cap. antec. erit

$$C=2B-A$$
$$D=2C-B$$
$$E=2D-C$$
$$F=2E-D$$

&c. &c.

Sit

Sit series algebrica secundi ordinis, cujus terminus generalis $A+Bn+Cn^2$; erit

$$D=3C-3B+A$$
$$E=3D-3C+B$$
$$F=3E-3D+C$$
$$G=3F-3E+D$$
&c. &c.

Sit series algebrica tertii ordinis, cujus terminus generalis $A+Bn+Cn^2+Dn^3$; erit

$$E=4D-6C+4B-A$$
$$F=4E-6D+4C-B$$
$$G=4F-6E+4D-C$$
$$H=4G-6F+4E-D$$
&c. &c.

Generatim igitur series algebricæ ordinis m sunt series recurrentes ordinis $m+1$. *Quod erat* &c.

691. Propositio III. *Si terminus generalis exprimens terminum* n^{esimum} *dicatur* T, *&* *summa terminorum per numerum* n *indicata vocetur* S, *deinde summa terminorum, quam designat numerus* n − 1, *fiat* = s; *erit* T = S − s.

Dem. Etenim si summæ terminorum per numerum $n-1$ impressæ addatur terminus n^{esimus} qui deficiebat, prodibit numerus integer n; ergo vicissim si a terminorum numero n subtrahatur summa terminorum, quos exprimit numerus $n-1$, redibit in residuo terminus n^{esimus}; At hic positus fuit terminus generalis (687.); ergo $T=S-s$. *Quod erat* &c.

692.

692. *Coroll.* Igitur si dato alicujus seriei termino summatorio expresso per functionem ipsius n, in hac functione pro n substituatur $n-1$, tum formula hinc genita a dato termino generali subtrahatur, residuum erit terminus generalis. Sic in dato termino summatorio $8n + \frac{1}{3}n + \frac{2}{3}n^3 = S$, si pro n substituatur $n-1$, proveniet $8n - 8 + \frac{1}{3}n - \frac{1}{3} + \frac{2}{3}n^3 - 2n^2 + 2n - \frac{2}{3} = s$; quare subtrahendo, fiet $S - s = 8 + \frac{1}{3} + 2n^2 - 2n + \frac{2}{3} = 9 - 2n + 2n^2 = T$, qui erit terminus generalis quaesitus.

693. *Scholion.* Tria hic sunt animadvertenda; (1.°) terminus generalis $S-s$ sub hac simplici forma non est admittendus, secus duae prodirent series, quae, si primus terminus primae & secundus secundae excipiantur, se mutuo destruerent. Sic si detur summa $S = \frac{n}{2+n}$, ponendo $n-1$ pro n, fiet $s = \frac{n-1}{n+1}$; hinc terminus generalis $S - s = \frac{n}{2+n} - \frac{(n-1)}{n+1}$; quapropter si haec formula non mutata remaneat, substitutis pro n successive numeris naturalibus 1, 2, 3, 4, &c. duae orirentur series (A), (B)

(A)

$$(A)\ldots\ldots \frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \frac{5}{7} \ldots\ldots\ldots \frac{n}{n+2}.$$

$$(B)\ldots\ldots \frac{0}{2}, -\frac{1}{3}, -\frac{2}{4}, -\frac{3}{5}, -\frac{4}{6} \ldots\ldots - \frac{(n-1)}{1+n}.$$

Quarum omnes termini præter ultimum in (A), & primum in (B) se elident, & summa evadet $\frac{n}{2+n} - \frac{0}{n}$, seu $\frac{n}{2+n}$, ex quo nihil colligitur. At si termini $\frac{n}{2+n} - \frac{(n-1)}{1+n}$ in aliam formam transmutentur, ipsos ad eundem denominatorem reducendo, ut fiat $S - s = \frac{2}{(1+n)(2+n)}$, series proveniet $\frac{2}{2\cdot3}, \frac{2}{3\cdot4}, \frac{2}{4\cdot5}, \frac{2}{5\cdot6}, \frac{2}{6\cdot7}$, &c. cujus summa $= \frac{n}{2+n}$. Quotiescumque igitur hoc artificium analyticum locum habet, utilis erit hæc methodus, secus inutilis.

(2.°) Ut obtineatur summa seriei in infinitum productæ, in summa generali cognita statuatur $n = \infty$. Sic in novissimæ seriei in infinitum productæ summa

ma $\frac{n}{2+n}$, ponendo $n=\infty$, fiet $\frac{n}{n}=1$, nam 2 præ ∞ evanefcit. Quod fi tunc n ad plures potentias affurgat, maxima ipfius poteftas, ceteris omnibus poteftatibus minoribus deletis, retineatur, quod omnes alii termini comparate ad illum, qui ad maximam potentiam affurgit, evadunt $=0$. Sic quum feriei jam exhibitæ 1, 7, 13, 19, &c. fumma generalis fit $3n^2 - n$, fi feries ponatur in infinitum continuata, fumma erit $3\infty^2 - \infty = 3\infty^2$.

(3.°) Quamvis haberi poffit $T = S - s$, non femper S eft exacta feriei fumma, poteft enim accidere, ut ipfa S differat a vera fumma per quantitatem aliquam conftantem, quæ ab n non dependeat. Ut id certo fcias, fubftitue 1 pro n tam in termino generali T, quam in fumma S; quod fi exinde prodeat $T=S$, erit S vera fumma; fic enim agendo fumma S ad valorem reducitur primi feriei termini, qui vi factæ fubftitutionis exprimi quoque debet ab ipfo termino generali T, adeoque fiet $T=S$. Quamobrem fi fuerit $T>S$, vel $T<S$, quoniam in utroque cafu non eft S vera fumma, ut talis evadat, ei eft in primo cafu addenda differentia inter T, & S, in fecundo vero demenda.

Sic quoniam $\frac{6n-3}{2} = \frac{3n^2-1}{2} - \frac{(3(n-1)^2-1)}{2}$,

fi pro n fubftituas 1, evadet $\frac{6n-3}{2} = T = \frac{3}{2}$,

sed $\frac{3n^2-1}{2}=S=1$; ergo T excedit S per $\frac{1}{2}$, quæ differentia si addatur ipsi S, summa exacta erit $\frac{3n^2-1}{2}-\frac{1}{2}=\frac{3}{2}n^2$. Ex quo monemur, formulas illas, quæ terminum constantem involvunt independentem ab n, veras serierum summas non exhibere, quia ponendo $n=0$, summa seriei non evanescet, ut fieri oportet, quare formulis hisce prætermissis, illæ tantummodo venient deinceps considerandæ, quarum unusquisque terminus continet n, numerum scilicet terminorum, ex quibus seriei propositæ summa debet emergere.

694. Propositio IV. *Invenire formulas indeterminatas exprimentes non tantum terminos generales, sed summas quoque generales serierum recurrentium algebricarum.*

Resolutio. Ordior ab illa serie, cujus summa generalis designatur per An. Pro n substituo $n-1$, deinde quidquid prodit subtraho ab A, & habeo $An-An+A=A$; hinc $T=S-s=A$, adeoque series ab hoc termino generali progenita terminis æqualibus constabit, eritque A, A, A, &c. Progredior ad illas series, quarum summa generalis est $An+Bn^2$. Substituendo de more $n-1$ pro n, invenio $T=S-s=An+Bn^2-An+A-Bn^2+2Bn-B=A-B+2Bn$; quare si pro n sufficiantur numeri naturales 1, 2, 3, 4, &c. nascetur series

ries $A+B, A+3B, A+5B, A+7B$, &c. feries nimirum algebrica primi ordinis, cujus primæ differentiæ conſtantes ſunt $2B$. Ergo in ſeriebus algebricis primi ordinis terminus generalis eſt $A-B+2Bn$, & ſumma generalis exprimetur per $An+Bn^2$.

Proponatur nunc ſumma generalis $An+Bn^2+Cn^3$. Facta ſolita ſubſtitutione quantitatis $n-1$ pro n, provenit $T=S-s=An+Bn^2+Cn^3-An+A-Bn^2+2Bn-B-Cn^3+3Cn^2-3Cn+C=A-B+C+2Bn-3Cn+3Cn^2$. In hoc termino generali ſubrogentur pro n numeri naturales $1, 2, 3, 4$, &c. & exurget ſeries $A+B+C, A+3B+7C, A+5B+19C$, &c. ſeries nempe ſecundi ordinis, in qua differentiæ ſecundæ conſtantes ſunt $6C$. Ergo in ſeriebus algebricis ſecundi ordinis formula termini generalis eſt $A-B+C+(2B-3C)n+3Cn^2$, & ſumma generalis repræſentatur per $An+Bn^2+Cn^3$.

Detur ſumma generalis $An+Bn^2+Cn^3+Dn^4$. Suffecto $n-1$ in locum ipſius n, reperietur $T=S-s=A-B+C-D+(2B-3C+4D)n+(3C-6D)n^2+4Dn^3$. Subſtituendo igitur pro n numeros $1, 2, 3, 4$, &c. prodibit ſeries algebrica tertii ordinis, videlicet $A+B+C+D, A+3B+7C+15D, A+5B+19C+65D, A+7B+37C+175D$, &c. in qua tertiæ differentiæ evadunt conſtantes, ſuntque $24D$. Ergo in ſeriebus algebricis tertii ordinis habetur terminus generalis $T=A-B+C-D+(2B-3C+4D)n+(3C-6D)n^2+4Dn^3$, & $S=An+Bn^2+Cn^3+Dn^4$.

Ex allatis igitur formulis indeterminatis colligitur, seriem algebricam cujuslibet ordinis m habere pro termino generali functionem quantitatis indeterminatæ n, in qua functione unus terminus est constans, & reliqui successivis ipsius n potentiis afficiuntur, incipiendo a potentia unitatis usque ad potentiam exponentis m; summam vero generalem esse functionem ejusdem n, in qua nullus terminus est constans, sed omnes successivis potentiis ipsius n sunt præditi, incipiendo a potentia unitatis usque ad potentiam exponentis $m+1$. Quamobrem terminus generalis seriem ordinis m designans exprimetur per formulam indeterminatam $A+Bn+Cn^2$

$+Dn^3 \ldots\ldots + Pn^m$, & summa generalis per $An+$

$Bn^2+Cn^3+Dn^4 \ldots\ldots + P^{m+1}$; *Quod erat* &c.

695. Propositio V. *Invenire indeterminatarum* A, B, C, *&c. in formulis nuper allatis existentium valores, ex quorum in propositis terminis* T, *ac summis generalibus* S *substitutione tam ipsi termini, quam summæ generales ad usum revocari possint.*

Resolutio. Quoniam in seriebus primas differentias constantes habentibus, vidimus, esse $T=A-B+2Bn$, & $S=An+Bn^2$ (694.), sumpta primi ordinis serie algebrica a; $a+b$; $a+2b$; $a+3b$; &c. positaque $n=1$, formulæ hinc prodeunti $T=A+B$ adæquo primum terminum $a=A+B$. Statuo præterea $n=2$, & $T=A+3B$ adæquo secundo seriei adsumptæ termino $a+b$, & obtineo $A+3B=a+b$. Ex hac æquatione primam $A+B$ $=a$

$= a$ ſubtraho, & oritur $2B = b$, adeoque $B = \frac{1}{2}b$. Subſtituo hunc valorem in $A + B = a$, fitque $A + \frac{1}{2}b = a$, ac proinde $A = a - \frac{1}{2}b$; ſubrogo valores hoſce tam in termino generali $T = A - B + 2Bn$, quam in ſumma generali $S = An + Bn^2$, & habeo

$$T = a - b + bn = a + \frac{(n-1)b}{1}$$

$$S = an - \frac{1}{2}bn + \frac{1}{2}bn^2 = an + \frac{n(n-1)b}{1.\ 2}.$$

Haud abſimilem operationem inſtituo in ſeriebus ſecundas differentias conſtantes habentibus, capta ſecundi ordinis ſerie algebrica a; $a + b$; $a + 2b + c$; $a + 3b + 3c$; $a + 4b + 6c$; $a + 5b + 10c$; &c. cujus ſecunda differentia conſtans c. Quum enim terminus generalis T ad iſtas ſeries pertinens ſit $A - B + C + (2B - 3C)n + 3Cn^2$, atque S ſit $= An + Bn^2 + Cn^3$; facta $n = 1$, primam efformo æquationem $A + B + C = a$; deinde poſita $n = 2$, ſecundam $A + 3B + 7C = a + b$; tum facta $n = 3$, tertiam $A + 5B + 19C = a + 2b + c$. Harum trium æquationum primam ſubtraho a ſecunda, deinde ſecundam a tertia, & duæ mihi obveniunt æquationes $2B + 6C = b$; $2B + 12C = b + c$; quarum primam demo a ſecunda, & habeo $6C = c$, ſeu $C = \frac{1}{6}c$. Subſtituo

hunc valorem in $2B+6C=b$, & prodit $B=\frac{1}{2}b-\frac{1}{2}c$, valoribusque ipsarum A, B in æquatione $A+B+C=a$ substitutis, fit $A=a-\frac{1}{2}b+\frac{1}{3}c$. Jam vero valores istos ipsarum A, B, C subrogo tam in termino, quam in summa generali, & habeo

$$T=a-b+c+bn-\frac{3}{2}cn+\frac{1}{2}cn^2=a+\frac{(n-1)}{1}b+\frac{(n-1)(n-2)}{1.\ 2}c.$$

$$S=an-\frac{1}{2}bn+\frac{1}{3}cn+\frac{1}{2}bn^2-\frac{1}{2}cn^2+\frac{1}{6}cn^3=$$

$$an+\frac{n(n-1)}{1.\ 2}b+\frac{n(n-1)(n-2)}{1.\ 2.\ 3}c.$$

Si calculum consimilem applices seriebus tertii ordinis, tertias nimirum differentias constantes habentibus (hujusmodi formulas generales mox exhibebimus) invenies

$$T=a+\frac{(n-1)}{1}b+\frac{(n-1)(n-2)}{1.\ 2}c+\frac{(n-1)(n-2)(n-3)}{1.\ 2.\ 3}d.$$

$S=$

$$S = an + \frac{n(n-1)}{1.\ 2} b + \frac{n(n-1)(n-2)}{1.\ 2.\ 3} c + \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4} d.$$

Ubi modus progrediendi tam quoad termini, quam quoad summæ generalis expressiones in reliquis seriebus ordinum subsequentium statim conspicitur; quare formulæ ad series quarti ordinis attinentes erunt

$$T = a + \frac{(n-1)}{1} b + \frac{(n-1)(n-2)}{1.\ 2} c + \frac{(n-1)(n-2)(n-3)}{1.\ 2.\ 3} d + \frac{(n-1)(n-2)(n-3)(n-4)}{1.\ 2.\ 3.\ 4} e.$$

$$S = n + \frac{n(n-1)}{1.\ 2} b + \frac{n(n-1)(n-2)}{1.\ 2.\ 3} c + \frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4} d + \frac{n(n-1)(n-2)(n-3)(n-4)}{1.\ 2.\ 3.\ 4.\ 5} e.$$

& sic deinceps. *Quod erat* &c.

696. *Corollarium.* Quoniam in adsumpta serie litterali algebrica cujuscumque ordinis primus terminus est a, prima differentia b, secunda c, tertia d, atque ita porro, si detur series algebrica numeris constans, primus ejusdem terminus stabit in adlatis formulis generalibus pro a, prima differentia pro b, secunda pro c, tertia pro d, & sic de reliquis.

Exemplum I. Esto series 2, 9, 24, 50, 90....&c. cujus tertiæ differentiæ constantes sunt 3. Ut valores litterarum a, b, c, d obtineantur in formulis ad series istas attinentibus, sumantur propositæ seriei differentiæ successivæ usque ad differentiam constantem sic

2, 9, 24, 50, 90 .... &c.
7, 15, 26, 40
8, 11, 14
3 3

eritque, vi Corollarii præcedentis, $a = 2$, $b = 7$, $c = 8$, $d = 3$; adeoque $T = 2 + \frac{(n-1)}{1} 7 + \frac{(n-1)(n-2)}{1 \cdot 2} 8 + \frac{(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3} 3$.

$$S = 2n + \frac{n(n-1)}{1 \cdot 2} 7 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} 8 + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} 3.$$

Exemplum II. Esto series 0, 10, 68, 234, 592, 1250, &c. quartas differentias constantes habens. Quum hujus primus terminus sit 0, erit $a = 0$; ut autem reliquarum litterarum b, c, d, e valores habeantur, accipiantur seriei differentiæ successivæ, ut in Corollario antecedente sic

0, 10, 68, 234, 592, 1250, &c.
10, 58, 166, 358, 658
48, 108, 192, 300
60, 84, 108
24, 24

qua-

quare $b = 10$, $c = 48$, $d = 60$, $e = 24$; factiſque ut ſupra ſubſtitutionibus in formulis idoneis, tam terminus generalis, quam ſumma generalis nullo negotio innoteſcent.

Exemplum III. Eſto ſeries $1^m, 2^m, 3^m, 4^m, 5^m$, &c. Si $m = 1$, numeri erunt naturales; ſi $m = 2$, ſeries erit algebrica ſecundi ordinis; ſi $m = 3$, ſeries erit algebrica tertii ordinis, & ſic deinceps. Quamobrem quum numerorum naturalium terminus generalis ſit n, quadratorum n^2, cuborum n^3, &c.; ſi Sn indicet ſummam numerorum naturalium, Sn^2 ſummam quadratorum, Sn^3 ſummam cuborum, atque ita porro, ſequens exurget tabella

$$Sn = \frac{n^2}{2} + \frac{n}{2}$$

$$Sn^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$Sn^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}$$

$$Sn^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$$

$$Sn^5 = \frac{n^6}{6} + \frac{n^5}{2} + \frac{n^4}{12} - \frac{n^2}{12}$$

&c. &c.

Sed de hiſce ſummis alibi generalius.

697.

697. *Scholion.* Quamvis numeri qui *figurati*, & *polygoni* appellantur, progreſſionibus arithmeticis, five algebricis alicujus ordinis ſint adnumerandi, adeoque ſub cenſum cadant ſerierum, quas haƈtenus pertraƈtavimus, attamen quum eorum cognitio ſit Tironibus neceſſaria, de his etiam ſeorſum agere Analyſtarum more conſtituimus, eo magis quod per ipſos expreſſiones jam traditæ ad terminos generales, ac ſummas pertinentes confirmantur. Sint igitur

Definitiones.

698. Series unitatibus conſtans, nimirum 1, 1, 1, 1, 1, &c. dicitur ſeries numerorum *figuratorum primi ordinis*.

699. Series numerorum naturalium 1, 2, 3, 4, 5, &c. vocatur ſeries numerorum *figuratorum ſecundi ordinis*.

700. Si in iſta accipiantur ſummæ duorum primorum terminorum $1 + 2$, trium priorum $1 + 2 + 3$, quatuor priorum $1 + 2 + 3 + 4$, & ſic deinceps, hæ ſummæ novam ſingillatim efficient ſeriem 1, 3, 6, 10, 15, 21, &c. quæ appellatur ſeries *figuratorum tertii ordinis*, ſeu numerorum *triangularium*.

701. Pari modo ſumptis ſummis terminorum hujus ſeriei, orietur ſeries *figuratorum quarti ordinis*, videlicet 1, 4, 10, 20, 35, &c. qui numeri *pyramidales* nuncupantur, & ita porro.

En hujuſmodi ſerierum proſpeƈtum.

1. ord. .....1, 1, 1, 1, 1, 1, 1, 1, 1, 1, &c. Num. *conſtantes*
2. ....1, 2, 3, 4, 5, 6, 7, 8, 9, 10, &c. ......*naturales*
3. ....1, 3, 6, 10, 15, 21, 28, 36, 45, ..&c. ....*triangulares*
4. ....1, 4, 10, 20, 35, 56, 84, &c. ....*pyramidales*
5. ....1, 5, 15, 35, 70, 126, &c....*triangulo-pyramidales*

&c.

702.

702. Numeri polygoni parum abſimilem geneſim ſortiuntur. Tres primas ſeries iſti habent communes cum numeris figuratis, ſed exordio facto a tertia ſerie 1, 3, 6, 10, 15, &c. erit unitas primus quartæ ſeriei numerus; ſumma primi, & ſecundi, nempe $1+3=4$ erit ſecundus numerus; ſumma ſecundi, & tertii, nimirum $3+6=9$, dabit tertium numerum, &c. quæ quarta ſeries dicitur ſeries numerorum *quadratorum*.

703. Pro quinta ſerie, præmiſſa unitate, cape ſummam ſecundi numeri quartæ ſeriei, & primi tertiæ, hoc eſt $4+1=5$, & exurget numerus ſecundus quintæ ſeriei; tum accipe ſummam tertii numeri quartæ ſeriei, & numeri ſecundi primæ ſeriei, ſcilicet $9+3=12$, qui erit numerus tertius quintæ ſeriei, &c. ac formabitur ſeries numerorum, qui *pentagoni* appellantur.

704. Pro ſexta ſerie præmiſſa de more unitate, ſummantur ſummæ numerorum quintæ ſeriei, ac tertiæ, prout egimus in quinta ſerie concinnanda, & ſexta proveniet ſeries numerorum, qui vocantur *hexagoni*, & ſic deinceps.

En numerorum polygonorum exordium.

Triangulares	1, 3, 6, 10, 15, 21, &c.
Quadrati	1, 4, 9, 16, 25, 36, &c.
Pentagoni	1, 5, 12, 22, 35, 51, &c.
Hexagoni	1, 6, 15, 28, 45, 66, &c.
&c.	&c.

Nos hìc de numeris tantummodo figuratis agemus.

705. Propoſitio VI. *Numerorum figuratorum ſummam, terminumque generalem exhibere.*

Re-

Resolutio. Seriei 1, 2, 3, 4, 5, &c. posita ut ante summa $= An + Bn^2$, ac statuendo $n = 1$, habebitur primus seriei terminus, nimirum $A + B = 1$. Si fiat $n = 2$, habebitur summa duorum primorum terminorum, hoc est $2A + 4B = 3$. Habemus igitur duas æquationes $A + B = 1$, $2A + 4B = 3$. Subtrahe duplum primæ a secunda, remanebit $B = \frac{1}{2}$; quare $A = 1 - B = \frac{1}{2}$. Hinc summa seriei propositæ $= \frac{1}{2} n + \frac{1}{2} n^2 = \frac{n(n+1)}{1 \cdot 2}$.

Accedat secunda series, cujus summa ponatur $= An + Bn^2 + Cn^3$. Eodem ratiocinio tres obtinebimus æquationes $A + B + C = 1$; $2A + 4B + 8C = 4$; $3A + 9B + 27C = 10$; quas per subtractionem reducemus ad duas, nempe $2B + 6C = 2$; $6B + 24C = 7$; ex quibus per novam subtractionem eliciemus $C = \frac{1}{6}$, $B = \frac{1}{2}$, $A = \frac{1}{3}$, ac proinde summa quæsita erit $= \frac{1}{3} n + \frac{1}{2} n^2 + \frac{1}{6} n^3 =$

$$\frac{n^3 + 3n^2 + 2n}{6} = \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}.$$

Pari modo reliquæ summæ invenientur prout extant in sequenti tabella cum respectivis terminis generalibus, qui obtinentur, si a singulis summis ultimus terminus dematur.

Ter-

706.

Termini generales	Ord.	*Series*	*Summæ*
1	1	1, 1, 1, 1, 1, 1, &c.	$\frac{n}{1}$
$\frac{n}{1}$	2	1, 2, 3, 4, 5, 6, &c.	$\frac{n(n+1)}{1.\ 2}$
$\frac{n(n+1)}{1.\ 2}$	3	1, 3, 6, 10, 15. 21, &c.	$\frac{n(n+1)(n+2)}{1.\ 2.\ 3}$
$\frac{n(n+1)(n+2)}{1.\ 2.\ 3}$	4	1, 4, 10, 20, &c.	$\frac{n(n+1)(n+2)(n+3)}{1.\ 2.\ 3.\ 4}$
$\frac{n(n+1)(n+2)(n+3)}{1.\ 2.\ 3.\ 4}$	5	1, 5, 15, &c.	$\frac{n(n+1)(n+2)(n+3)(n+4)}{1.\ 2.\ 3.\ 4.\ 5}$
&c.		&c.	&c.

706. *Coroll.* 1. Hinc seriei figuratorum ordinis m summa =

$$\frac{n(n+1)(n+2)(n+3)(n+4)\ldots(n+m-1)}{1.\ 2.\ 3.\ 4.\ 5.\ m};$$

Terminus generalis =

$$\frac{n(n+1)(n+2)(n+3)\ldots\ldots(n+m-2)}{1.\ 2.\ 3.\ 4.\ (m-1)}.$$

707. *Coroll.* 2. Quoniam vero $S-s=$

$$\frac{n(n+1)(n+2)\ldots\ldots(n+m-1)}{1.\ 2.\ 3.\ m} -$$

$$\frac{(n-1)n(n+1)(n+2)\ldots\ n+m-2}{1.\ 2.\ 3.\ 4.\ m}$$; stabit

$$\frac{n(n+1)(n+2)(n+3)\ldots(n+m-1)}{1.\ 2.\ 3.\ 4.\ m} -$$

$$\frac{(n-1)n(n+1)(n+2)\ldots\ldots(n+m-2)}{1.\ 2.\ 3.\ 4.\ m} =$$

$$\frac{n(n+1)(n+2)\ldots\ldots(n+m-2)}{1.\ 2.\ 3.\ (m-1)}.$$ Sic si fuerit

$n=10, m=6$, erit $\frac{10}{1}\cdot\frac{11}{2}\cdot\frac{12}{3}\cdot\frac{13}{4}\cdot\frac{14}{5}\cdot\frac{15}{6}-\frac{9}{1}\cdot$

$\frac{10}{2}\cdot\frac{11}{3}\cdot\frac{12}{4}\cdot\frac{13}{5}\cdot\frac{14}{6}=\frac{10}{1}\cdot\frac{11}{2}\cdot\frac{12}{3}\cdot\frac{13}{4}\cdot\frac{14}{5}$, quod est verissimum.

708.

708. Propoſitio VII. *Data ſumma, ac termino generali ſeriei attinentis ad numeros figuratos ordinis* m, *hujus ſeriei expreſſionem invenire.*

Reſolutio. In termino generali pone ſucceſſive $n = 1, 2, 3, 4$, &c. & ſeriei quæſitæ terminos gradatim obtinebis. Sic primus terminus erit

$$\frac{1.2.3.4.5\ldots\ldots(m-1)}{1.2.3.4.5\ldots\ldots(m-1)} = 1 \text{ ; ſecundus } =$$

$$\frac{2.3.4\ldots\ldots m}{1.2.3\ldots\ldots(m-1)} = m \text{ ; tertius}$$

$$\frac{3.4.5.6\ldots\ldots(m+1)}{1.2.3.4\ldots\ldots(m-1)} = \frac{m(m+1)}{1.\ 2.} \text{ ; quartus}$$

$$= \frac{m(m+1)(m+2)}{1.\ 2.\ 3} \text{ ; \&c. quare ſeries quæſita}$$

erit $1, \frac{m}{1}, \frac{m(m+1)}{1.\ 2}, \frac{m(m+1)(m+2)}{1.\ 2.\ 3.}$,

$\frac{m(m+1)(m+2)(m+3)}{1.\ 2.\ 3.\ 4}$, &c. cujus patet terminum generalem eſſe

$$\frac{m(m+1)(m+2)(m+3)(m+4)(m+5)(m+6)\ldots(m+n-2)}{1.\ 2.\ 3.\ 4.\ 5.\ 6.\ 7.\ \ldots n-1}.$$

Sed ſupra habuimus terminum generalem ejuſdem ſeriei ſub alia expreſſione, videlicet

$$\frac{n(n+1)(n+2)(n+3)\ldots\ldots(n+m-2)}{1.\ 2.\ 3.\ 4\ldots\ldots\ldots(m-1)} \text{ ; ergo}$$

for-

formabitur æquatio $\frac{m(m+1)(m+2)(m+3)\ldots m+n-2}{1.\ 2.\ 3.\ 4\ldots\ldots(n-1)}$

$= \frac{n(n+1)(n+2)(n+3)\ldots\ldots(n+m-2)}{1.\ 2.\ 3.\ 4\ldots\ldots\ldots\ldots(m-1)}$,

quicumque ſint numeri m, n; ſive, ponendo $p = m - 2$, erit $\frac{(p+2)(p+3)(p+4)(p+5)\ldots(p+n)}{1.\ 2.\ 3.\ 4.\ (n-1)} =$

$\frac{n(n+1)(n+2)(n+3)\ldots\ldots(n+p)}{1.\ 2.\ 3.\ 4\ldots\ldots\ldots(p+1)}$. Sic ſi fuerit $p = 10, n = 6$, ſtabit $\frac{12.13.14.15.16}{1.2.3.4.5} =$

$\frac{6.7.8.9.10.11.12.13\ 14.15.16}{1.2.3.4.\ 5.\ 6.\ 7.\ 8.\ 9.10.11}$. *Quod erat* &c.

709. Propoſitio VIII. *Formulas generales progreſſionum arithmeticarum ſeu algebricarum, cujuſcumque ordinis, nec non earum terminum generalem, ac ſummam rurſus exhibere.*

Reſolutio. Diſponantur numeri figurati, ut in ſequenti tabella

Ter-

Termini generales	Series	Summæ
1	1, 1, 1, 1, 1, 1, 1, 1..	n
$\frac{n-1}{1}$	0, 1, 2, 3, 4, 5, 6, 7..	$\frac{n(n-1)}{1.\ 2}$
$\frac{(n-1)(n-2)}{1.\ 2}$	0, 0, 1, 3, 6, 10, 15, 21..	$\frac{n(n-1)(n-2)}{1.\ 2.\ 3}$
$\frac{(n-1)(n-2)(n-3)}{1.\ 2.\ 3}$	0, 0, 0, 1, 4, 10, 20, 35..	$\frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4}$
$\frac{(n-1)(n-2)(n-3)(n-4)}{1.\ 2.\ 3.\ 4}$	0, 0, 0, 0, 1, 5, 15, 35..	$\frac{n(n-1)(n-2)(n-3)(n-4)}{1.\ 2.\ 3.\ 4.\ 5}$
$\frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{1.\ 2.\ 3.\ 4.\ 5}$	0, 0, 0, 0, 0, 1, 6, 21..	$\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1.\ 2.\ 3.\ 4.\ 5.\ 6}$
&c.	&c.	&c.

Et quoniam secundæ columnæ horizontali addita est una cyphra o, adeoque aucta una figura; si n designet totas dictæ columnæ figuras, ut quantitatum T, S expressiones in Prop. præc. adlatæ hic accommodari possint, ab ipsa n unitas demi debet, quare erit $T = n - 1$, $S = \frac{n[n-1]}{1 \,.\, 2}$. Eadem ratione in termino generali, ac summa tertiæ columnæ horizontalis, ob auctas duas cyphras o, pro n scribi debet $n-2$, ac proinde erit $T = \frac{[n-1][n-2]}{1 \,.\, 2}$, $S = \frac{n[n-1][n-2]}{1 \,.\, 2 \,.\, 3}$, & sic de ceteris, ut schema demonstrat. Hoc determinato ac fixo, tam prima columna horizontalis, quam ejus terminus generalis, & summa multiplicentur per a. Pari modo secunda multiplicetur per b, tertia per c, quarta per d, &c. deinde fiat summa duarum priorum, tum trium, præterea quatuor, &c. ex quo prodit.

(1.) a, a, a, a, a, a, &c. $T = a$; $S = na$.

(2.) a, $a+b$, $a+2b$, $a+3b$, &c. $T = a + \frac{(n-1)}{1}b$; $S = na + \frac{n(n-1)}{1 \,.\, 2 \,.}b$.

(3.) a, $a+b$, $a+2b+c$, $a+3b+3c$, $a+4b+6c$, &c. $T = a + \frac{(n-1)}{1}b + \frac{(n-1)(n-2)}{1 \,.\, 2}c$; $S = na$

+

$+\frac{n(n-1)}{1.\ 2}b+\frac{n(n-1)(n-2)}{1.\ 2.\ \ 3}c.$

(4.) $a, a+b, a+2b+c, a+3b+3c+d, a+4b+6c+4d, a+5b+10c+10d; a+5b+15c+20d$, &c.

$$T = a+\frac{(n-1)}{1}b+\frac{(n-1)(n-2)}{1.\ \ 2}c+$$

$$\frac{(n-1)(n-2)(n-3)}{1.\ \ 2.\ \ 3}d.$$

$$S = na+\frac{n(n-1)}{1.\ 2}b+\frac{n(n-1)(n-2)}{1.\ 2.\ \ 3}c+$$

$$\frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ \ 3.\ \ 4}d.$$

&c.

Numerorum (2.), (3.), (4) &c. seriebus perpensis, reperientur series arithmeticæ primi, secundi, tertii &c. ordinis, termini vero generales & summæ cum jam expositis (695.) egregie conspirant; *Quod erat* &c.

710. *Coroll.* Si series in adnexa Tabella sic procederent

1, 2, 3. 4, 5, 6, &c.
0, 1, 3. 6, 10, 15, &c.
0, 0, 1, 4, 10, 20, &c.
0, 0, 0, 1, 5, 15, &c.
&c.

tunc quod una cyphra 0 a singulis columnis hori-

zontalibus ablata fuit, pro n (inverse agendo) substitui debet $n+1$ in singulis terminis generalibus, qui proinde evaderent n; $\frac{n[n-1]}{1.\ 2}$; $\frac{n[n-1][n-2]}{1.\ 2.\ 3}$; $\frac{n[n-1][n-2][n-3]}{1.\ 2.\ 3.\ 4}$, &c.
Sed illæ columnæ horizontales sunt columnæ verticales efformatæ a coëfficientibus potestatum binomii [67]; ergo hujusmodi coëfficientium formulæ generales per numeros quoque figuratos inveniuntur.

Pro numerorum figuratorum usu in generalibus coëfficientium terminis exprimendis additur sequens.

711. Propositio IX. *Duos numeros invenire ejus conditionis, ut secundus dividat primum tot vicibus, quot unitates insunt in* p *cum residuo; hoc vero residuum dividat secundum totidem vicibus* p *cum alio residuo; hoc item secundum residuum dividat primum residuum totidem vicibus* p, *atque ita porro, donec divisio fiat sine residuo.*

Resolutio. Numerus primus esto $=x$, secundus $=y$; primum residuum $=z$, secundum $=v$, tertium $=t$, quartum $=s$, quintum $=q$, sextum $=r$, septimum &c. sequentes nascentur æquationes $\frac{x}{y}=p+\frac{z}{y}$; $\frac{y}{z}=p+\frac{v}{z}$; $\frac{z}{v}=p+\frac{t}{v}$; $\frac{v}{t}=p+\frac{s}{t}$; $\frac{t}{s}=p+\frac{r}{s}$; $\frac{s}{r}=p+\frac{q}{r}$; $\frac{r}{q}=p$

+

$+ \frac{n}{q}$; $\frac{q}{n} = p$, &c.

Ex his æquationibus, inversè procedendo, infertur $q = pn$; $r = pq + n = p^2n + n$; $s = pr + q = p^2q + pn + q = p^3n + 2pn$; $t = ps + r = p^2r + pq + pq + n = p^4n + p^2n + 2p^2n + n = p^4n + 3p^2n + n$; $v = pt + s = p^5n + 3p^3n + pn + p^3n + 2pn = p^5n + 4p^3n + 3pn$; $z = pv + t = p^6n + 4p^4n + 3p^2n + p^4n + 3p^2n + n = p^6n + 5p^4n + 6p^2n + n$; $y = pz + v = p^7n + 5p^5n + 6p^3n + pn + p^5n + 4p^3n + 3pn = p^7n + 6p^5 + 10p^3n + 4pn$; $x = py + z = p^8 + 6p^6n + 10p^4n + 4p^2n + p^6n + 5p^4n + 6p^2n + n = p^8 + 7p^6n + 15p^4n + 10p^2n + n$. Quas æquationes si separatim colliges, habebis

1. pn
2. $p^2n + n$
3. $p^3n + 2pn$
4. $p^4n + 3p^2n + n$
5. $p^5n + 4p^3n + 3pn$
6. $p^6n + 5p^4n + 6p^2n + n$
7. $p^7n + 6p^5n + 10p^3n + 4pn$
8. $p^8n + 7p^6n + 15p^4n + 10p^2n + n$
9. $p^9n + 8p^7n + 21p^5n + 20p^3n + 5pn$
10. $p^{10}n + 9p^8n + 28p^6n + 35p^4n + 15p^2n + n$

&c.

Si harum quantitatum naturam contemplor, reperio (1.°) quod si primus ipsius p exponens in columnis horizontalibus ponatur $= m$, secundus erit $m - 2$, tertius $m - 4$, quartus $m - 6$, & sic deinceps; (2.°) quod omnes termini sunt per n multiplicati;

 (3.°)

(3.°) quod in secunda columna verticali coëfficientes sunt numeri figurati secundi ordinis, seu naturales, quorum terminus generalis $= m - 1$; (4.°) quod in tertia columna verticali coëfficientes sunt numeri figurati tertii ordinis, seu triangulares, quorum terminus generalis $= \frac{[m-2][m-3]}{1 \;.\; 2}$; (5.°) quod quartæ columnæ coëfficientes sunt figurati quarti ordinis, seu pyramidales, quorum terminus generalis $= \frac{[m-3][m-4][m-5]}{1.\;2.\;3}$, atque ita porro. Quare formula generalis erit

$$11.\; p^m n + [m-1]p^{m-2}n + \frac{[m-2][m-3]}{1 \;.\; 2}p^{m-4}n$$

$$+ \frac{[m-3][m-4][m-5]}{1.\;2.\;3}p^{m-6}n +$$

$$\frac{[m-4][m-5][m-6][m-7]}{1.\;2.\;3.\;4}p^{m-8}n +$$

$$\frac{[m-5][m-6][m-7][m-8][m-9]}{1.\;2.\;3.\;4.\;5}p^{m-10}n +$$

&c. nec non

$$12.\; p^{m-1}n + [m-2]p^{m-3}n + \frac{[m-3][m-4]}{1 \;.\; 2}p^{m-5}n$$

$$+$$

$$+ \frac{[m-4]\,[m-5]\,[m-6]}{1.\quad 2.\quad 3} p^{m-7} n + \frac{[m-5][m-6][m-7][m-8]}{1.\quad 2.\quad 3.\quad 4} p^{m-9} n + \text{\&c.}$$

Itaque si duo dentur numeri inæquales, quorum minor majorem tot vicibus dimetiatur, quot insunt unitates in p cum residuo, & hoc residuum dimetiatur numerum minorem totidem vicibus p sine residuo, duo isti numeri erunt illi, quos per 1, 2 a tergo designavimus. Quod si tot hujusmodi divisiones sine residuo effici debeant, duo isti numeri erunt per 3, 2 notati. Si decem esse debeant tales divisiones, hi duo numeri per 10, 9 erunt designati, & sic deinceps; *Quod erat* &c.

712. *Coroll.* Si ponatur $m = \infty$, duo numeri quæsiti erunt infiniti; at duæ inveniri poterunt quantitates finitæ, quæ sint invicem, ut duo isti numeri infiniti. Sic si statuatur $p = 1$, seceturque recta quævis in extrema & media ratione, recta integra continebit semel portionem majorem cum residuo minoris portionis; major autem portio minorem continebit cum residuo, & ita semper in infinitum.

Transeundum nunc est ad series algebricas fractas.

713. Propositio X. *Datis summis generalibus indeterminatis, invenire terminos generales in seriebus terminos fractos habentibus, ita ut tam fractionum numeratores, quam denominatores, series algebricas inter se constituant.*

Resolutio I. Esto primum summa generalis $\frac{Ln}{A+Bn}$.

Per jam adlatas rationes pro n pono $n-1$, eritque

$$T = S - s = \frac{Ln}{A+Bn} - \frac{L[n-1]}{A+B[n-1]} =$$

$\frac{AL}{\overline{A+B[n-1]} \times \overline{A+Bn}}$. Hujus formulæ numerator est constans, denominator vero duobus coalescit factoribus, quorum unusquisque velut terminus generalis consideratus seriem exhibet primi ordinis, ubi eadem adest differentia B, hoc tantum cum discrimine, quod secunda series incipit a secundo termino primæ. Sic si pro n successive substituantur numeri naturales $1, 2, 3, 4$, &c. in terminis $A+B[n-1]$, $A+Bn$, duæ series emergent

$A, A+B, A+2B, A+3B$, &c.

$A+B, A+2B, A+3B$, &c.

quarum secunda incipit a secundo termino primæ.

II. Esto secundo summa generalis

$\frac{Ln+Mn^2}{\overline{A+B(n-1)} \times \overline{A+Bn}}$. Substituendo ut supra

$n-1$ pro n, prodit $\frac{L(n-1)+M(n-1)^2}{\overline{A+B(n-2)} \times \overline{A+B(n-1)}}$.

Duæ istæ fractiones eundem habent divisorem $A+B(n-1)$, quare si primæ numerator multiplicetur per $A+B(n-2)$, & secundæ numerator per

$A+$

$A+Bn$, ambæ ad idem nomen redigentur, fietque

$$T=S-s=\frac{\begin{matrix}AL-BLn\\ AM-BMn\\ +2AMn\end{matrix}}{\overline{A+B(n-2)}\times\overline{A+B(n-1)}\times\overline{A+Bn}}.$$

In hac formula numerator est terminus generalis primi ordinis, dummodo n non evanescat, secus formula in quantitatem constantem migraret. Denominator vero constat tribus factoribus seriem primi ordinis singulis exhibentibus, ubi eadem est differentia, sed ita, ut series secunda incipiat a secundo termino primæ, ac tertia a tertio, ut perspicuum fiet, tribuendo successive quantitati n numeros 1, 2, 3, 4, &c.

III. Pari methodo invenientur termini generales serierum, quarum summæ generales sint

$$\frac{Ln+Mn^2+Nn^3}{\overline{A+B(n-2)}\times\overline{A+B(n-1)}\times\overline{A+Bn}};$$

$$\frac{Ln+Mn^2+Nn^3+Pn^4}{\overline{A+B(n-3)}\times\overline{A+B(n-2)}\times\overline{A+B(n-1)}\times\overline{A+Bn}};\ \&c.$$

Quod si detur summa generalis, cujus denominator sit aliqua functio ipsius n, veluti $\frac{Ln+Mn^2}{A+Bn+Cn^2}$, substituendo $n-1$ pro n, habetur

$T=$

$$T = \frac{\begin{matrix} AL + 2AMn + BMn^2 \\ - AM - BMn - CLn^2 \end{matrix}}{\overline{A + B(n-1) + C(n-1)^2} \times \overline{A + Bn + Cn^2}};$$

& sic de ceteris; *Quod erat* &c.

714. *Coroll.* Si termini generalis numerator sit per aliquem ex ultimis factoribus illum componentibus divisibilis, peracta divisione, pertinebit ad terminum ordinis proxime inferioris.

715. *Monitum* I. Cum unus vel plures factores intermedii in denominatore deficiunt, oportet multiplicare numeratorem, ac denominatorem termini generalis per factores deficientes, ita ut denominator debitum possideat factorum numerum. Sic aderunt in denominatore series requisitæ, & secunda incipiet a secundo termino primæ, tertia a tertio, & sic deinceps. Ex. gr. si terminus generalis inventus sit $\frac{P}{\overline{A+B(n-3)} \times \overline{A+Bn}}$, quum factores qui in denominatore desiderantur, sint $A+B(n-2)$, $A+B(n-1)$, terminus generalis completus erit

$$\frac{P \times \overline{A+B(n-2)} \times \overline{A+B(n-1)}}{\overline{A+B(n-3)} \times \overline{A+B(n-2)} \times \overline{A+B(n-1)} \times \overline{A+Bn}}.$$

Similiter si detur series $\frac{1}{2.11}$, $\frac{1}{5.14}$, $\frac{1}{8.17}$, $\frac{1}{11.20}$, &c. quoniam in secundo termino factor 5 incipere debet (*ex hyp.*) a secundo numero denomi-

minatoris primi termini, primus ex duobus mediis arithmeticis numerorum 2, 11, erit 5, ac proinde secundus erit 8; ergo primus dictæ seriei terminus evadet $\frac{5 \cdot 8}{2 \cdot 5 \cdot 8 \cdot 11}$. Pari ratione quum in tertio seriei termino primus denominatoris numerus incipi debeat a tertio numero denominatoris primi termini, primus medius arithmeticus inter factores 5, & 14, erit 8, adeoque secundus seriei terminus fiet $\frac{8 \cdot 11}{5 \cdot 8 \cdot 11 \cdot 14}$, & ita porro, quare proposita series ad hanc methodum accommodata erit $\frac{5 \cdot 8}{2 \cdot 5 \cdot 8 \cdot 11}$,

$$\frac{8 \cdot 11}{5 \cdot 8 \cdot 11 \cdot 14}, \frac{11 \cdot 14}{8 \cdot 11 \cdot 14 \cdot 17}, \frac{14 \cdot 17}{11 \cdot 14 \cdot 17 \cdot 20}, \&c.$$

716. *Monitum* II. Si in dati termini generalis numeratore exponens ipsius n sit æqualis vel major numero factorum in denominatore existentium, factoribus istis invicem ductis, per factum dividatur numerator; sic terminus generalis in duas formulas, alteram integram, fractam alteram abrumpetur, quare summa primæ per superius acta (695.) invenietur, secundæ vero prout infra docebitur. Excipe casum, in quo maximus ipsius n exponens in termini generalis numeratore sit una tantum unitate minor numero factorum denominatorem efformantium, tunc enim methodus deficit, & summa

quam

quam dicta methodus exhibet, non est vera seriei summa, quod probe animadvertendum. Sin autem memoratus exponens sit duabus saltem unitatibus minor quam numerus factorum ipsius denominatoris, summa algebricæ seriei obtineri semper poterit.

717. Propositio XI. *Formulas generales pro summis serierum algebricarum terminos fractos habentium, simul cum indeterminatarum* A, B, C, *&c. valoribus concinnare.*

Resolutio I. Sumpto summæ generalis indeterminatæ $\frac{Ln}{A+Bn}$ termino generali invento

$$\frac{AL}{\overline{A+B(n-1)} \times \overline{A+Bn}}$$, series, quæ a primo factore $A+B(n-1)$ progignitur, fit $a, a+b, a+2b, a+3b$, &c. series, quæ (ponendo $n=1,2,3$, &c.) oritur a secundo factore $A+Bn$, erit $a+b$, $a+2b, a+3b, a+4b$, &c. Ut quantitatum A, B valores per primum terminum a, & differentiam b determinentur, in factore $A+B(n-1)$ pro n substitue 1, deinde facta cum respondentis seriei primo termino comparatione, stabit $A=a$. Suffice deinde 2 pro n, & cum secundo dictæ seriei termino fiat comparatio; habebitur $A+B=a+b$. Ex hac secunda æquatione primam $A=a$ subtrahe, & obtinebis $B=b$. Tum facta constante $AL=c$, valores hosce subroga in termino generali, qui evadet

c

$\frac{c}{a+(n-1)b \times \overline{a+bn}}$. Hinc ob $AL = aL = c$, erit $L = \frac{c}{a}$, adeoque ſumma generalis $\frac{Ln}{A+Bn} = \frac{cn}{a(a+bn)}$.

II. Tranſeo ad terminum generalem

$$\frac{\begin{matrix} AL - BLn \\ -AM - BMn \\ +2AMn \end{matrix}}{\overline{A+B(n-2)} \times \overline{A+B(n-1)} \times \overline{A+Bn}},$$

cujus numerator quum ſit & ipſe terminus generalis ſeriei primi ordinis, dummodo non evadat $n = 0$, ſeriei algebricæ primi ordinis hinc genitæ primus terminus vocetur c, & differentia d; habebitur $c+(n-1)d = \begin{matrix} AL - BLn \\ -MA - BMn \\ +2AMn \end{matrix}$ (178.). Si vero ſeries primo denominatoris factori $A+B(n-2)$ reſpondens ſit a, $a+b$, $a+2b$, $a+3b$, &c.; ſeries, quæ ſecundo $A+B(n-1)$ reſpondet, erit $a+b$, $a+2b$, $a+3b$, &c., & ſeries tertio $A+Bn$ reſpondens erit $a+2b$, $a+3b$, $a+4b$, &c. Pro valoribus A, B ut ſupra determinandis, poſita $n=1$, ſtabit $A+B(n-2) = A-B = a$. Deinde ſtatuendo $n=2$, fiet $A+B(n-2) = A = a+b$; quo valore in $A-B = a$ ſuffecto, prodit $B = b$. Subſtituantur iſti valores in termino generali; proveniet

$T =$

$$T = \frac{c+(n-1)d}{\overline{a+b+(n-2)b} \times \overline{a+b+(n-1)b} \times \overline{a+b+bn}}.$$

Ut summa generalis $\dfrac{Ln+Mn^2}{\overline{A+B(n-1)} \times \overline{A+Bn}}$ per easdem litteras determinetur, quoniam faciendo $n=1$, tam in termino, quam in summa generali, primus seriei terminus emergit, pono ex utraque parte $n=1$, tum sumpto denominatoris

$\overline{A+B(n-1)} \times \overline{A+Bn}$ valore $\overline{a+b+(n-1)b}$

$\times \overline{a+b+bn}$, qui propter $n=1$ evadit $(a+b)$ $(a+2b)$, comparo terminum generalem cum summa generali, & habeo $\dfrac{L+M}{(a+b)(a+2b)} =$

$\dfrac{c}{a(a+b)(a+2b)}$, sive $L+M=\dfrac{c}{a}$.

Quum autem statuendo $n=2$, summa generalis præbeat duos primos seriei terminos, sumo in termino generali duos primos seriei terminos, substituendo in ipsa pro n, primo 1, deinde 2, ex quo habeo in primo casu $\dfrac{c}{a(a+b)(a+2b)}$, in secundo $\dfrac{c+d}{(a+b)(a+2b)(a+3b)}$; quorum ad idem no-

nomen reductorum summa $= \frac{a(c+d)+(a+3b)c}{a(a+b)(a+2b)(a+3b)}$, hanc adæquo summæ generali, in qua $n=2$, & provenit

$$\frac{2L+4M}{(a+2b)(a+3b)} = \frac{a(c+d)+c(a+3b)}{a(a+b)(a+2b)(a+3b)},$$

sive $L+2M = \frac{a(c+d)+c(a+3b)}{2a(a+b)}$, ex qua si subducatur $L+M=\frac{c}{a}$, remanet $M=$

$$\frac{a(c+d)+c(a+3b)}{2a(a+b)} - \frac{c}{a} = \frac{c(b-a)+a(c+d)}{2a(a+b)}.$$

Hoc autem valore suffecto in æquatione $L+M= \frac{c}{a}$, obtinetur valor $L=\frac{c(3a+b)-a(c+d)}{2a(a+b)}$. Si demum valores inventi quantitatum A, B, L, M, &c. subrogentur in summa generali indeterminata, prodibit

$$S=\frac{\overline{c(3a+b)-a(c+d)}\times n+\overline{c(b-a)+a(c+d)}\times n^2}{2a(a+b)\times\overline{a+b(n-1)}\,b\times\overline{a+b+bn}}.$$

III. Consimili methodo reliquæ formulæ tam terminorum, quam summarum generalium invenientur, iisdem litteris cum earum significatione retentis. Sic pro seriebus, quarum summa generalis

$$\frac{Ln + Mn^2 + Nn^3}{\overline{A+B(n-2)} \times \overline{A+B(n-1)} \times \overline{A+Bn}},$$

habetur $T =$

$$\frac{c + \frac{(n-1)}{1} d + \frac{(n-1)(n-2)}{1 \cdot 2} e}{\overline{a+2b+(n-3)b} \times \overline{a+2b+(n-2)b} \times \overline{a+2b+(n-1)b} \times \overline{a+2b+bn}};$$

$$S = \Big(\overline{a^2(6c-3d+2e)+ab(12c-4d+2e)+4b^2c} \times n +$$

$$\overline{3a^2(d-e)+3ab(2c+d-e)+6bc} \times n^2 +$$

$$\overline{a^2e+ab(d+e)+2b^2c} \times n^3 \Big):$$

$$6a\overline{(a+b)(a+2b)(a+2b+(n-2)b)} \times$$

$$\overline{a+2b+(n-1)b} \times \overline{a+2b+bn}.$$

Quod si in serie proposita terminorum numerator sit constans, tam in termino T, quam in summa S, omnes termini litteras d, e, &c. includentes deleantur.

IV. Deveniamus ad terminum generalem

$$\frac{AL + 2AMn + BMn^2 - AM - BMn - CLn^2}{\overline{A+B(n-1)+C(n-1)^2} \times \overline{A+Bn+Cn^2}}.$$

Quum in hoc numerator sit terminus generalis seriei algebricæ secundi ordinis, si hujus seriei primus terminus fiat $= \beta$, prima differentia $= \delta$, secunda (quæ est constans) $= \varepsilon$, numerator ille poterit exprimi per $\beta + \frac{(n-1)}{1}\delta + \frac{(n-1)(n-2)}{1 \cdot 2}\varepsilon$ (709.).

Præ-

Præterea quoniam denominator duobus constat factoribus, quorum unusquisque est terminus generalis seriei algebricæ secundi ordinis; si primæ seriei primus terminus ponatur $= a$, prima differentia $= b$, secunda (quæ est constans) $= c$, secundæ seriei primus terminus $= p$, prima differentia $= q$, secunda erit $= b$, series enim istæ æqualem habent secundam differentiam constantem, quia in hoc tantum differunt inter se, quod secunda incipit a primæ secundo termino. Iste igitur denominator designari poterit per $\overline{a + \frac{(n-1)}{1}b + \frac{(n-1)(n-2)}{1 \, . \, 2}c} \times \overline{p + \frac{(n-1)}{1}q + \frac{(n-1)(n-2)}{1 \, . \, 2}c}$, adeoque pro termino generali stabit

$$T = \frac{\beta + \frac{(n-1)}{1}\delta + \frac{(n-1)(n-2)}{1 \, . \, 2}\varepsilon}{\overline{a + \frac{(n-1)}{1}b + \frac{(n-1)(n-2)}{1 \, . \, 2}c} \times \overline{p + \frac{(n-1)}{1}q + \frac{(n-1)(n-2)}{1 \, . \, 2}c}}.$$

Nunc est summa generalis per easdem litteras definienda. Itaque tam in termino T, quam in summa generali $\frac{Ln + Mn^2}{A + Bn + Cn^2}$, statuo $n = 1$, atque ista fit $\frac{L + M}{p}$, quod $A + Bn + Cn^2 = p + \frac{(n-1)}{1}q + \frac{(n-1)(n-2)}{1 \, . \, 2}c$; tum facta termini T cum summa

generali comparatione prodit $L + M = \frac{\beta}{a}$. Præterea ponendo in termino T primo $n = 1$, deinde $n = 2$, ac ſumendo aggregatum, obtineo $\frac{\beta}{ap} + \frac{\beta + \delta}{(a+b)(p+q)} = \frac{\beta(a+b)(p+q) + ap(\beta+\delta)}{ap(a+b)(p+q)}$.
Ex hac formula efficio juxta methodum num. III. æquationem in termino generali T, ponendo $n = 2$, & quia tunc fit $A + Bn + Cn^2 = p + q$, oritur $L + 2M = \frac{\beta(a+b)(p+q) + ap(\beta+\delta)}{2ap(a+b)}$; quare $M =$

$$\frac{\beta(a+b)(p+q) + ap(\beta+\delta)}{2ap(a+b)} - \frac{\beta}{a} =$$

$$\frac{\beta(a+b)(p+q) + ap(\beta+\delta) - 2\beta p(a+b)}{2ap(a+b)},$$ ac proinde $L = \frac{4\beta p(a+b) - \beta(a+b)(p+q) - ap(\beta+\delta)}{2ap(a+b)}$.
Ergo ſubſtituendo, ſumma generalis quæſita erit

$$S = \frac{\left.\begin{array}{l} \overline{4\beta p(a+b) - \beta(a+b)(p+q) - ap(\beta+\delta)} \times n + \\ \overline{\beta(a+b)(p+q) + ap(\beta+\delta) - 2\beta p(a+b)} \times n^2 \end{array}\right\}}{2ap(a+b) \times \overline{p + \frac{(n-1)}{1} q + \frac{(n-1)(n-2)}{1 \cdot 2}} \times c.}$$

Se-

Series, quæ ad novissimum terminum generalem referuntur, duos habent pro denominatore factores, duo quadrata repræsentantes; numerator autem esse potest vel series secundi ordinis, vel series primi ordinis, vel etiam quantitas constans; in secundo casu fit $\varepsilon = 0$, in tertio $\delta = 0$, $\varepsilon = 0$. Methodus consimilis adhibetur in aliis seriebus relatis ad terminos generales, in quibus denominatoris factores exprimunt cubos; biquadrata &c. *Quod erat* &c.

Exemplum. I. Proponatur series

$$\frac{1}{4.7.10.13}, \frac{7}{7.10.13.16}, \frac{17}{10.13.16.19}, \frac{31}{13.16.19.22}, \text{\&c.}$$

quum horum terminorum denominatores quatuor factoribus constent, series ista ad num. III. pertinet, ubi termini generalis T denominator quatuor terminis coalescit. Sumantur interea seriei, quam numeratores efficiunt, differentiæ sic

$$\begin{array}{c} 1,\ 7,\ 17,\ 31,\ \text{\&c.} \\ 6,\ 10,\ 14 \\ 4\quad 4 \end{array}$$

primus terminus erit $1 = c$; prima differentia $6 = d$; secunda $4 = e$. Differentia, qua denominatorum series progrediuntur, est $3 = b$, & primus primi denominatoris terminus $4 = a$. Ergo substituendo fiet

$$T = \frac{1 + 6\frac{(n-1)}{1} + 4\frac{(n-1)(n-2)}{1.\quad 2}}{\overline{10+3(n-3)} \times \overline{10+3(n-2)} \times \overline{10+3(n-1)} \times 10+3n}$$

$$S = \frac{-76n + 294n^2 + 202n^3}{1680 \times \overline{10+3(n-2)} \times \overline{10+3(n-1)} \times 10+3n}$$

=

$$= \frac{-\frac{19}{420}n + \frac{7}{40}n^2 + \frac{101}{840}n^3}{(4+3n)(7+3n)(10+3n)};$$ quamobrem si ponatur $n = \infty$, peractis denominatoris multiplicationibus, ob n, n^2, &c. numerorum omnium præ n^3 evanescentiam, summa seriei propositæ in infinitum productæ erit $\frac{101\,n^3}{27 \cdot 3 \cdot 4 \cdot 5\,n^3} = \frac{101}{22680}$.

Exemplum II. Esto series ad num. IV. attinens

$$\frac{1}{2^2 . 5^2}, \frac{3}{5^2 . 8^2}, \frac{5}{8^2 . 11^2}, \frac{7}{11^2 . 14^2}, \frac{9}{14^2 . 17^2}, \text{\&c.}$$

Ergo $\varepsilon = 0$, adeoque $\beta = 1$, $\delta = 2$, quia primus numeratorum terminus est $= 1$, & prima differentia, quæ est constans, $= 2$. Quoad denominatores ex duobus factoribus conflatos, quum priorum prima series sit $2^2, 5^2, 8^2, 11^2$; &c., seu 4, 25, 64, 121, &c. capiantur primæ & secundæ differentiæ sic

4 , 25 , 64 , 121 , &c.
21 , 39 , 57 ,
18 , 18

eritque $a = 4$, $b = 21$, $c = 18$. Secundorum factorum series est 5^2, 8^2, 11^2, 14^2, &c. sive 25, 64, 121, 196, &c. cujus sumptis primis, ac secundis differentiis sic

25 , 64 , 121 , 196 , &c.
39 , 57 , 75
18 18

habetur $p = 25$, $q = 39$. Valoribus istis in formulis

lis tam terminum generalem quam summam exhibentibus suffectis, prodibit $T =$

$$\frac{1+2(n-1)}{\overline{4+21\frac{(n-1)}{1}+18\frac{(n-1)(n-2)}{1\,.\,2}} \times \overline{25+39\frac{(n-1)}{1}+18\frac{(n-1)(n-2)}{1\,.\,2}}}$$

$$S = \frac{12n+13n^2}{100 \times \overline{25-39\frac{(n-1)}{1}+18\frac{(n-1)(n-2)}{1\,.\,2}}}$$

Si vero ponatur $n = \infty$, vel si series in infinitum excurrat, summa erit $= \frac{13}{900}$.

718. Propositio XII. *Dato termino generali* T *expresso per functionem* $an^m + bn^{m-1} + cn^{m-2} + \&c.$ *in qua tot termini, quot libuerit, sint adsumpti, summam generalem* S *investigare.*

Resolutio. Posita $S = An^{m+1} + Bn^m + Cn^{m-1} + \&c.$ pro n substitue $n-1$, factaque formulæ hinc prodeuntis evolutione, habebis

$$\begin{aligned} s = An^{m+1} - A(m+1)n^m &+ Am\frac{(m+1)}{2}n^{m-1} - A\frac{(m+1)m(m-1)}{2\,.\,3}n^{m-2} + \&c. \\ &+ Bn^m - Bmn^{m-1} + Bm\frac{(m-1)}{2}n^{m-2} - \&c. \\ &+ Cn^{m-1} - C(m-1)n^{m-2} + \&c. \\ &+ Dn^{m-2} + \&c. \end{aligned}$$

quapropter $S - s = T = an^m + bn^{m-1} + cn^{m-2} + \&c. =$

$$A(m+1)n^{m} - \frac{1}{2}Am(m+1)n^{m-1} + \frac{1}{2}\frac{A(m+1)m(m-1)}{3}n^{m-2} - \&c.$$
$$+ \quad Bmn^{m-1} - \frac{1}{2}B\,m\,(m-1)\,n^{m-2} + \&c.$$
$$+ \quad C(m-1)\,n^{m-2} - \&c.$$

Terminisque homologis conlatis, invenietur $A = \frac{a}{m+1}$; $B = \frac{1}{2}a + \frac{b}{m}$; $C = \frac{c}{m-1} + \frac{1}{2}b + \frac{1}{12}am$, &c. ex quo prodit $S = \frac{a}{m+1}n^{m+1} + \left(\frac{b}{m} + \frac{1}{2}a\right)n^{m} + \left(\frac{c}{m-1} + \frac{1}{2}b + \frac{1}{12}am\right)n^{m-1}$ + &c. summæ nimirum quæsitæ expressio generalis casibus particularibus adplicanda. *Quod erat* &c.

Exemplum. Esto series $1^m + 2^m + 3^m + 4^m +$&c. cujus terminus generalis n^m. Quoniam primus terminus $a = 1$, $b = 0$, $c = 0$, seriei summa erit $S = \frac{1}{m+1}n^{m+1} + \frac{1}{2}n^m + \frac{1}{12}mn^{m-1} +$ &c. indicante n numerum terminorum, quem series exhibet.

Corollaria.

719. Hinc si $m = 1$, seriei finitæ $1+2+3+4$ + &c. summa erit rursus $= \frac{1}{2}n^2 + \frac{1}{2}n$, nam ubi quan-

quantitatis n exponens fit $=0$, ibi juxta formulæ Newtonianæ pro binomio ad poteſtatem indeterminatam elevato Regulam', eſt ſiſtendum. Sin autem fuerit $m=2$, ſeriei finitæ $1^2+2^2+3^2+4^2+$ &c. ſumma erit $=\frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n$, prorſus ut alibi; & ſic de reliquis.

720. Si fuerit $n=\infty$; tunc terminis n^m, n^{m-1}, &c. præ n^{m+1} evaneſcentibus, fiet $1^m+2^m+3^m+4^m+\ldots\ldots+\infty^m=\frac{\infty^{m+1}}{m+1}$, ubi quantitas infinita ∞ eſſe debet poſitiva, ſecus ſumma foret quantitas finita, excepto caſu, in quo $m=-1$.

721. Si $m=\frac{1}{r}$, erit ſeriei $1^{\frac{1}{r}}+2^{\frac{1}{r}}+3^{\frac{1}{r}}+4^{\frac{1}{r}}$ + &c. ſive $\sqrt[r]{1}+\sqrt[r]{2}+\sqrt[r]{3}+\sqrt[r]{4}+$ &c. ſumma $=\frac{r}{1+r}n^{\frac{1+r}{r}}+\frac{1}{2}n^{\frac{1}{r}}+\frac{1}{12}\frac{n}{r}^{\frac{1-r}{r}}+$ &c. $=$

$$\frac{r}{1+r}\sqrt[r]{n^{1+r}}+\frac{1}{2}\sqrt[r]{n}+\frac{1}{12r}\sqrt[r]{n^{1-r}}+\text{ \&c.}$$

Sed hæc ſumma eſt infinita; ergo fiet $=\frac{r\,\infty^{\frac{1+r}{r}}}{1+r}$ $=\frac{r}{1+r}\sqrt[r]{\infty^{1+r}}$.

722. Si m fuerit numerus negativus, eadem series constabit terminis itidem infinitis.

723. Si posito numero $P =$ seriei finitæ $1^m + 2^m + 3^m + 4^m +$ &c. fiat multiplicatio per p^n, proveniet $p^n P = p^n + 2^m p^n + 3^m p^n + 4^m p^n +$ &c. adeo ut si $p = 2$, $n = m$, foret $2^m P = 2^m + 4^m + 6^m + 8^m +$ &c.

724. *Scholion.* Si in functione $an^m + bn^{m-1} + cn^{m-2} +$ &c. plures termini pro libito unus post alium capiantur, ac deinde pro n substituantur in singulis successive numeri $1, 2, 3, 4,$ &c. variæ nascetur series algebricæ, quarum simul sumptarum summa generalis semper erit in promptu.

CAPUT IV.

DE SERIEBUS GEOMETRICIS, ET ALGEBRICO-GEOMETRICIS SUMMAM RECIPIENTIBUS.

Hypotheses.

725. Series geometrica *primi ordinis* exprimitur per K^n, vel generalius per AK^n, si enim pro n sumantur successive numeri $1, 2, 3, 4,$ &c. orietur series geometrica $AK, AK^2, AK^3, AK^4,$ &c. 726.

726. Series geometrica *secundi ordinis* designatur per $AK^n \pm BH^n$, nimirum per duas series geometricas; etenim hinc exurgit series AK, AK^2, AK^3
BH, BH^2, BH^3
&c. constans ex duabus seriebus geometricis, quarum termini homologi sint addendi, vel subtrahendi, hoc est AK, & BH; AK^2 & BH^2, &c. ita ut $AK \pm BH$ efficiat primum terminum seriei secundi ordinis; $AH^2 \pm BH^2$ terminum secundum, & ita porro.

727. Series geometrica *tertii ordinis* repræsentatur per $AK^n \pm BH^n \pm CI^n$, quod conflatur ex tribus progressionibus geometricis simul additis, vel subtractis, adhibita cautione velut in addendis, aut subtrahendis duabus seriebus secundi ordinis.

Generatim si terminorum $AK^n \pm BK^n \pm CI^n \pm DL^n \pm$ &c. numerus sit n, per ipsos designabitur series geometrica ordinis n.

728. *Definitio* I. Summa, quam recipiunt series geometricæ, ideo dicitur *exponentialis*, quod in formula serierum istarum summam exhibente litera n terminorum numerum designans locum tenet exponentis.

729. Propositio I. *Quantitatis constantis, puta* K *(denominatorem rationis designantis) elevatæ ad potestatem indeterminatam* n *vera summa non est* AK^n, *sed* $AK^n - A = S$.

Dem.

Dem. Pone $n-1$ loco n, ac fiet $AK^{n-1} - A = s$. Ergo $S - s = T = AK^n - A - (AK^{n-1} - A) = AK^n - AK^{n-1} = A(K-1)K^{n-1}$. Statue $n = 1$; erit $T = A(K-1) = AK - A = S$, ut fieri debet, etenim tam summa generalis, quam terminus generalis primum seriei terminum suppeditant. At id non accidit, si pro S sumas AK^n; ergo &c.

730. Propositio II. *In omni serie geometrica, tam termini* T, *quam summæ generalis* S *formulas exhibere.*

Resolutio. Per formulas in Prop. præced. traditas possem Problema resolvere, sed inverse. Ut id directe exequar, resolutionem ex jam prolatis (186. n. XII. & XVI.) derivare satius duco, formulas quæsitas cum supra adlatis conspirantes deinde demonstraturus. Itaque datis primo termino a, terminorum numero n, & denominatore rationis m, vidimus, terminum generalem esse $T = am^{n-1}$, summam generalem $S = \frac{a}{m-1}(m^n - 1)$ si $m > 1$, nimirum si series sit crescens, vel $\frac{a}{1-m}(1 - m^n)$ si $m < 1$, videlicet si series fuerit decrescens. Sed si in primo casu ponas $A = \frac{a}{m-1}$, $K = m$, invenies $A(K^n - 1) =$

$= \frac{a}{m-1}(m^n - 1) = S$ & $A(K-1)K^{n-1} = am^{n-1}$

$= T$; si vero in secundo casu ponas $A = \frac{a}{1-m}$, habebis $A(K^n - 1) = \frac{a}{1-m}(1 - m^n) = S$, & $A(K-1)$ $K^{n-1} = am^{n-1} = T$. *Ergo* &c.

Corollaria.

731. Si in serie crescente ponatur n infinita, erit K^n, seu m^n infinita, adeoque tam series, quam ejusdem summa infinitæ. Si in serie decrescente fiat n infinita, K^n, seu m^n evadet infinitesima, quare termino generali stante am^{n-1} summa erit $A = \frac{a}{1-m}$.

732. Si $K = 1$, omnes seriei termini, ejusque summa fit $= 0$. Si in serie crescente statuatur $a = 1$, & tam terminus, quam summa generalis per m multiplicentur, ille fiet $= m^n$, ista $= \frac{m^{n+1} - m}{m-1}$; Contra in serie decrescente, manente illo $= m^n$, ista fiet $\frac{m - m^{n+1}}{1-m}$.

733.

733. Si denominator rationis sit quantitas negativa, series cum signorum alternatione progredietur. Tunc autem vel primus progressionis terminus est positivus, & secundus negativus, vel contra. Si primum; series veluti decrescens considerari debet, quare prima formula $S = \frac{a}{1-m}(1-m^n)$ est adhibenda. Si secundum; series veluti crescens reputanda, adeoque altera formula $S = \frac{a}{m-1}(m^n - 1)$ in usum vocanda.

734. Quum series geometricæ altiorum ordinum nascantur ex aggregatis, vel differentiis serierum geometricarum primi ordinis (727.), earum tam termini quam summæ generales ex terminis & summis generalibus serierum generatricium (sumptis aggregatis, vel residuis) inveniuntur. Sic si series geometricæ primi ordinis sint A, B, &c. F, G, &c. & series proposita ea sit, quæ oriatur ex A, B, &c. $+F, -G$, &c. hujus seriei terminus generalis constabit terminis generalibus serierum A, B, &c. quibus additus sit terminus generalis seriei F, detracto deinde termino generali seriei G, &c. quod similiter de summa generali est intelligendum.

735. Quod si alias series ortas ex aggregato, vel residuo serierum geometricarum integrarum, vel fractarum, ordinis cujuslibet excogitare lubeat, eadem regula tam in terminis, quam in summis generalibus reperiundis locum obtinet.

Exem-

Exempla.

I. Esto series $1+2+4+8+16+32+64$, ita ut per septem hosce numeros sit $n=7$; quoniam $a=1$, $m=2$, hujusmodi numerorum summa

$\frac{a}{m-1}(m^n-1)=128-1=127.$

II. Esto series $\frac{b}{c}+\frac{b^2}{c^2}+\frac{b^3}{c^3}+\frac{b^4}{c^4}+$ &c. nec non alia series $\frac{e}{f}+\frac{e}{fg}+\frac{e}{fg^2}+\frac{e}{fg^3}+$ &c. quum in prima serie sit $a=\frac{b}{c}=m$, & in secunda $a=\frac{e}{f}$, $m=\frac{1}{g}$, ponendo in utraque $n=\infty$, primæ summa erit $\frac{b}{c-b}$ (731.), secundæ erit $\frac{eg}{f(g-1)}$; quamobrem ipsarum serierum in infinitum abeuntium aggregatum $=\frac{b}{c-b}+\frac{eg}{f[g-1]}$. Patet autem, terminum generalem in prima esse $\frac{b^n}{c^n}$, in altera $\frac{e}{f[g^{n-1}]}$.

III. Esto series geometrica crescens $3, 5, 8\frac{1}{3}, 13\frac{8}{9}, 23\frac{4}{27}$, &c. sumpta respondenti formula $S=$

$= \frac{a}{m-1}(m^n - 1)$, ob $a = 3$, $m = \frac{5}{3}$, substituendo, fiet $S = \frac{3^2}{2}\left(\frac{5^n}{3^n} - 1\right) = \frac{5^n - 3^n}{2 \cdot 3^{n-2}}$, &c. $T = am^{n-1} = \frac{3 \cdot 5^{n-1}}{3^{n-1}} = \frac{5^{n-1}}{3^{n-2}}$.

IV. Esto $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}$, &c. series nempe geometrica decrescens (735.), cui convenit formula $S = \frac{a}{1-m}(1 - m^n)$. Ob $a = \frac{1}{2}$, $m = -\frac{1}{2}$, erit $T = am^{n-1} = \frac{1}{2}\left(-\frac{1}{2}\right)^{n-1} = -\overline{1 \times -\tfrac{1}{2}} \times \left(-\frac{1}{2}\right)^{n-1} = -1 . \left(-\frac{1}{2}\right)^n$; atque $S = \frac{1}{3} . \overline{1 - \left(-\frac{1}{2}\right)^n}$; quare si ponatur $n = \infty$, erit $S = \frac{1}{3}$.

V. Esto series crescens $\frac{5}{2}, \frac{39}{4}, \frac{271}{8}$, &c. genita ex subtractione seriei (C) ab aggregato serierum (A), (B), nimirum

(A)

(*A*) . . . . . 2, 6, 18, 54, 162, &c.
(*B*) . . . . 1, 4, 16, 64, 256, &c.

(*C*) . . . . $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$, &c.

Ob $a = 2, m = 3$ in ſerie (*A*), ejus terminus generalis (ſubſtituendo) erit $= 2 \cdot 3^{n-1}$; ſumma generalis $= 3^n - 1$. Ob $a = 1, m = 4$ in ſerie (*B*), ejus terminus generalis erit $= 4^{n-1}$; ſumma generalis $= \frac{1}{3}(4^n - 1)$.

Ob $a = \frac{1}{2}, m = \frac{1}{2}$ in ſerie (*C*), ejus terminus generalis erit $= \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}$; ſumma generalis $= 1 - \left(\frac{1}{2}\right)^n$.

Ergo ſeriei propoſitæ terminus generalis $T = 2 \cdot 3^{n-1} + 4^{n-1} - \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}$.

Summa generalis $S = 3^n - 1 + \frac{1}{3}(4^n - 1) - 1$

+

$+\left(\frac{1}{2}\right)^n = 3^n + \frac{1}{3}4^n - \frac{7}{3} + \left(\frac{1}{2}\right)^n.$

736. Propositio III. *Dato seriei geometricæ termino generali, summam generalem invenire.*

Resolutio. In termino generali T proposito pone $n = 1$, habebis primum terminum a. Fac deinde $n = 2$, & quidquid T tunc evadet divide per a, quotus, seu denominator rationis erit $= m$; quibus valoribus in $S = \frac{a}{m-1}(m^n - 1)$ suffectis, summæ generalis valor quæsitus emerget. *Quod erat* &c.

Exemplum. Detur terminus generalis $T = \frac{4^{3n-1}}{2^{2n+1}}$. Ut summæ S valorem habeas, in T pone $n = 1$, ac fiet $\frac{4^2}{2^3} = \frac{16}{8} = 2 = a$; tum in eodem T statue $n = 2$, eritque $\frac{4^5}{2^5} = 32$, quo per $2 = a$ diviso, prodit $\frac{32}{2} = 16 = m$. Ergo ob $m > 1$, peractis in $S = \frac{a}{m-1}(m^n - 1)$ substitutionibus, habetur $S = \frac{2}{15}(16^n - 1)$.

737.

737. Propositio IV. *Series geometricæ cujuscumque ordinis per formulam* $AK^n + BH^n + CI^n +$ &c. *expressæ, sunt series recurrentes ejusdem ordinis.*

Dem. Quoad seriem geometricam primi ordinis res patet, quilibet enim terminus per K multiplicatus sequentem exhibet. Seriei geometricæ secundi ordinis quilibet terminus ex duobus antecedentibus conflatur; etenim si terminus secundus, sive termino quæsito propior multiplicetur per $K+H$, & primus per $-KH$, nascetur terminus expetitus. Sic si terminus iste fuerit $AK^n + BH^n$, ille, qui hunc statim præit, erit $AK^{n-1} + BH^{n-1}$, & alter huic proximus $AK^{n-2} + BH^{n-2}$; quare si $K+H$ ducatur in $AK^{n-1} + BH^{n-1}$, atque $-KH$ in $AK^{n-2} + BH^{n-2}$, fiatque summa, emerget $AK^n + BH^n$. In serie geometrica tertii ordinis tres termini antecedentes subsequentem suppeditant. Sit enim terminus quæsitus generaliter expressus per $AK^n + BH^n + CI^n$; termini, qui ordine inverso ipsum præcedunt, multiplicentur singuli sequenti modo;

$AK^{n-1} + BH^{n-1} + CI^{n-1}$ per $K+H+I$.

$AK^{n-2} + BH^{n-2} + CI^{n-2}$ per $-(KH+KI+HI)$.

$AK^{n-3} + BH^{n-3} + CI^{n-3}$ per KHI;

prodibit $AK^n + BH^n + CI^n$, si multiplicatione exacta fiat summa. Quod si fuerit series geometrica quarti ordinis, cujus terminus generalis $AK^n + BH^n + CI^n + DL^n$; quilibet ejus terminus exurget ex summa quatuor præcedentium, quorum ultimus ductus sit in $K + H + I + L$, penultimus in $-(KH+KI+KL+HI+HL+IL)$, antepenultimus in $KHI + KHL + HIL$, & qui antepenultimum, præit in $-KHIL$. Sic prosequendo demonstrabitur, series quoque geometricas altiorum ordinum ad series recurrentes ejusdem ordinis semper esse referendas; *Quod erat* &c.

738. *Coroll.* 1. Ex hoc igitur criterio cujus generis, vel ordinis sit series geometrica proposita, dignosci potest.

739. *Coroll.* 2. Quoniam seriei recurrentis ordinis n quilibet terminus oritur ex aggregato terminorum præcedentium, quorum numerus n, ita ut ultimus sit multiplicatus per numerum n quantitatum K, H, I, &c. penultimus per binarum hujusmodi quantitatum summam præeunte signo contrario; antepenultimus per ternarum summam, & ita semper cum signorum alternatione; liquet, quantitates istas terminum propositum per multiplicationem efformantes eadem cum æquationis cujuslibet coëfficien-

cientibus proprietate gaudere. Quamobrem quantitates K, H, I, &c. quarum numerus m, erunt radices æquationis gradus m. Igitur si ponamus $K + H + I +$ &c. $= \alpha$; $-(KH + KI + KL +$ &c.$) =$ β; $KHI + KHL +$ &c. $= \gamma$; $- KHIL$ &c. $= \delta$; &c.

æquatio pro radicibus habens K, H, I, &c. erit x^n

$- \alpha x^{n-1} + \beta x^{n-2} - \gamma x^{n-3} + \delta x^{n-4} -$ &c. $= 0$, adeoque per hanc æquationem quantitates K, H, I, &c. definientur. Pro quantitatibus vero A, B, C, &c. (quæ omnes adsumuntur inæquales) determinandis requiruntur series algebrico-geometricæ, ad quas proinde contemplandas gradum facimus.

740. *Definitio* II. Si seriei algebricæ $A + Bn +$

$Cn^2 + Dn^3 + \ldots\ldots Z^m$ termini omnes multiplicentur per singulos seriei geometricæ terminos, series hinc genitæ vocantur *algebrico-geometricæ*.

741. Propositio V. *Cujuscumque seriei algebrico-geometricæ tam terminum, quam summam generalem, indagare*.

Resolutio. Esto $(A + Bn)K^n - A$. Quum hæc formula constet serie algebrica primi ordinis ducta in seriem geometricam pariter primi ordinis, ejus terminus generalis componetur ex termino generali primæ seriei per terminum generalem seriei secundæ multiplicato; sed terminus generalis primæ seriei ex dictis est $a + \frac{(n-1)}{1} b$, terminusque generalis secundæ est pm^{n-1}, posito termino primo $= 1$, nu-

 me-

mero terminorum $= n$, & denominatore rationis $= m$. Ergo formulæ $(A + Bn) K^n - A$ terminus generalis $= a + \frac{(n-1)}{1} bpm^{n-1}$. Pari ratione si in seriebus arithmeticis secundi ordinis primus terminus sit $= a$, prima differentia $= b$, secunda $= c$, formulæ $(A + Bn + Cn^2) K^n - A$ terminus generalis erit $(a + \frac{n-1}{1} b + \frac{n-1 . n-2}{1 . 2} c) \times pm^{n-1}$; & sic de ceteris; *Quod erat* &c.

742. Propositio V. *Serierum algebrico-geometricarum summæ habent pro termino generali formulam algebricam ejusdem gradus ductam in eandem quantitatem exponentialem.*

Dem. Esto summa $(A + Bn) K^n - A$. Si pro n ponatur $n - 1$, novamque formulam hinc prodeuntem a prima de more subtrahatur, prodibit terminus generalis $(A + Bn) K^n - (A - Bn + B) K^{n-1}$
$= \frac{(A(K-1) + B + (K-1) Bn) K^n}{K}$.

Itidem ex summa $(A + Bn + Cn^2) K^n - A$ elicitur terminus generalis

AK

$$\frac{\left.\begin{array}{l} AK + BKn + CKn^2 \\ -A \\ +B - Bn \\ -C + 2Cn - Cn^2 \end{array}\right\} K^n}{K}.$$

Prosequere quousque libet, ac semper propositionis veritas constabit, subtractio enim (ut ex necessario calculi progressu patet) neque ad nihilum redigere valet quantitatem n, neque ipsius potestates immutare. *Quod erat* &c.

743. Propositio VI. *Sumpto seriei algebrico-geometricæ termino generali indeterminato* $(e + fn + gn^2 + hn^3 + \&c.)K^n = t$, *summæ generalis formulas pro quolibet casu particulari concinnare.*

Resolutio. Posito dicto termino generali, accipe summam generalem $(A + Bn + Cn^2 + Dn^3 + \&c.)K - A = S$ ejusdem gradus cum adsumpto termino generali. Ex hac summa erue, ut moris est, terminum generalem T. Hunc compara cum jam adsumpto termino generali t, qui eidem T est æqualis, & valores indeterminatarum A, B, C, &c. expressos per e, f, g, &c. substitue in summa generali S. Ista in casibus particularibus (subrogando pro K, e, f, g, &c. valores datos) summam quæsitam largietur, dummodo advertatur, quod si fuerit $K < 1$, signa tam in termino generali T, quam in summa S mutentur. *Quod erat* &c.

744. *Coroll.* 1. Terminus generalis datus esto $(e + fn)K^n$, sive $(e + fn)m^n$, statuendo ut ante

 m

m pro K In formula generali summæ $[A+Bn]m^n - A = S$, quæ ipsi termino generali respondet, pro n scribe $n-1$, factaque ut supra novæ formulæ a prima subtractione, comperies $T =$

$$\frac{[A[m-1]+B+B[m-1]n]}{m}m^n = [e+fn]m^n .$$

Comparentur termini homogenei; fiet $\frac{Bn[m-1]}{m}$

$= fn$, adeoque $B = \frac{fm}{m-1}$; $\frac{A[m-1]+B}{m} = e$,

unde $A = \frac{em-B}{m-1} = \frac{em}{m-1} - \frac{fm}{[m-1]^2}$; quibus valoribus in summæ generalis formula $[A+Bn]m^n - A$ suffectis, provenit $S = \left[\frac{em}{m-1} - \frac{fm}{(m-1)^2} + \frac{fmn}{m-1}\right]m^n - \left[\frac{em}{m-1} - \frac{fm}{(m-1)^2}\right]$.

Quod si fuerit $m < 1$, ad evitandas quantitates negativas mutentur signa tam in termino, quam in summa generali, adeoque erit $T =$

$$\frac{[A[1-m]-B+B[1-m]n]}{m}m^n; \; S = A -$$

$[A+Bn]m^n$, ubi advertendum, quod si fuerit n $=$

$=\infty$, erit $[A+Bn]m^n$ quantitas infinite parva, ac proinde feriei in infinitum protractæ summa $=A$. Jam vero formularum T, t terminis homologis rursus comparatis, invenitur $B=\frac{fm}{1-m}$; $A=\frac{em}{1-m}+\frac{fm}{[1-m]^2}$; quibus valoribus substitutis, obtinetur

$$S=\frac{em}{1-m}+\frac{fm}{[1-m]^2}-\left[\frac{em}{1-m}+\frac{fm}{[1-m]^2}+\frac{fn}{1-m}\right]m^n.$$

Exempla.

I. Si terminus generalis fuerit $[7+2n]\,3^n$; erit $m=3$; $e=7$; $f=2$; unde $S=\left[\frac{21}{2}-\frac{6}{4}+\frac{6n}{2}\right]3^n-\frac{21}{2}-\frac{6}{4}=[9+3n]\,3^n-9=[3+n]3^{n+1}-3^2$.

II. Terminus generalis sit $\left[-\frac{2}{3}+\frac{3}{4}n\right]\frac{1}{5^n}$.

Erit $m=\frac{1}{5}$; $e=-\frac{2}{3}$; $f=\frac{3}{4}$. Hinc $\frac{em}{1-m}+\frac{fm}{[1-m]^2}=\left[-\frac{2}{3\cdot 5}+\frac{3}{4^2}\right]:\frac{4}{5}=\frac{-2}{3\cdot 4}+$

$\frac{3 \cdot 5}{4^3} = \frac{-2 \cdot 4^2 + 3^2 \cdot 5}{3 \cdot 4^3} = \frac{13}{3 \cdot 4^3}$; $\frac{fmn}{1-m} = \frac{3n}{4^1}$.

Quamobrem $S = \frac{13}{3 \cdot 4^3} - \frac{1}{5^n}\left[\frac{13}{3 \cdot 4^3} + \frac{3n}{4^2}\right]$;

ſerieque in infinitum continuata, $S = \frac{13}{3 \cdot 4^3}$.

III. Duarum, ſerierum $1, 2, 3, 4, 5, 6$, &c.

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$ &c.

ſingulis terminis invicem ductis, generetur nova ſeries $\frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \frac{5}{32}, \frac{6}{64}$, &c. Adverto, primam habere terminum generalem $= n$, alteram $= \frac{1}{2^n}$. Novæ igitur hinc natæ ſeriei terminus generali erit $\frac{n}{2^n}$; quare ob $m = \frac{1}{2} < 1$, ſumpta ſecund formula generali, reperietur $m = \frac{1}{2}$, $e = 0$; $f = 1$; ex quo $S = 2 - \frac{1}{2^n}(2 + n) = \frac{2^{n+1} - 2 - n}{2^n}$.

745. *Coroll.* 2. Si ſeriei algebrico-geometricæ ſumma exprimatur per formulam $(A + Bn + Cn^2)n^s -$

I

$A = S$, terminus generalis s erit $= (e + fn + gn^2)m^n$, & vicissim; quare statuendo $n - 1$ pro n, novamque formulam a prima S subtraendo, fiet

$$\frac{\left.\begin{array}{l} Am + Bmn + Cmn^2 \\ -A \\ +B \quad -Bn \\ -C \quad +2Cn - Cn^2 \end{array}\right\}}{m} = e + fn + gn^2,$$

adeoque instituta de more comparatione, habetur $\frac{Cm - C}{m} = g$, & $C = \frac{gm}{m-1}$; $\frac{Bm - B + 2C}{m} = f$, & $B = \frac{fm - 2C}{m-1} = \frac{fm}{m-1} - \frac{2gm}{(m-1)^2}$; $\frac{Am - A + B - C}{m} = e$, & $A = \frac{em - B + C}{m-1} =$

$$\frac{em}{m-1} - \frac{fm}{(m-1)^2} + \frac{2gm}{(m-1)^3} + \frac{gm}{(m-1)^2}.$$

Substitue hos valores in summa S, & emerget $S =$

$$\left[\frac{em}{m-1} - \frac{fm + gm}{(m-1)^2} + \frac{2gm}{(m-1)^3} + \overline{\frac{fm}{m-1} - \frac{2gm}{(m-1)^2}} \times n + \frac{gm}{m-1}n^2\right]m^n - \left[\frac{em}{m-1} - \frac{fm+gm}{(m-1)^2} + \frac{2gm}{(m-1)^3}\right].$$

Hæc

Hæc formula valet cum $m > 1$; cum autem $m < 1$, mutatis signis, provenit

$$S = \frac{em}{1-m} + \frac{fm-gm}{(1-m)^2} + \frac{2gm}{(1-m)^3} - \Big[\frac{em}{1-m} + \frac{fm-gm}{(1-m)^2} + \frac{2gm}{(1-m)^3} + \overline{\frac{fm}{1-m} + \frac{2gm}{(1-m)^2}} \times n + \frac{gm}{1-m}n^2\Big] m^n.$$

Exempla.

I. Esto $(1+n+n^2)2^n = T$. Erit $m = 2$; $e = 1$; $f = 1$; $g = 1$. Hinc $S = (6 - 2n + 2n^2)2^n - 6 = 2\,\overline{(3 - n + n^2)2^n - 3}$.

II. Esto series algebrica secundi ordinis $\frac{3}{2}, \frac{9}{2}, \frac{18}{2}, \frac{30}{2}, \frac{45}{2}$, &c. cujus differentiæ secundæ $= \frac{3}{2}$; nec non series geometrica $2, 3, \frac{9}{2}, \frac{27}{4}, \frac{81}{8}$, &c. in qua denominator rationis $= \frac{3}{2}$. Harum serierum singulis terminis invicem ductis, oritur nova series $\frac{3\cdot 2}{2}, \frac{9\cdot 3}{2}, \frac{18\cdot 9}{2\cdot 2}, \frac{30\cdot 27}{2\cdot 4}, \frac{45\cdot 81}{2\cdot 8}$, &c. Quum au-

autem terminus generalis primæ sit $= \frac{3}{4}n + \frac{3}{4}n^2$, & alterius $= 2 \cdot \left(\frac{3}{2}\right)^{n-1}$, terminus generalis novæ seriei erit $= \left(\frac{3}{4}n + \frac{3}{4}n^2\right) 2 \cdot \left(\frac{3}{2}\right)^{n-1} = (n+n^2)\left(\frac{3}{2}\right)^n$. Ergo ob $m > 1$, sumpta prima formula generali, stabit $m = \frac{3}{2}$; $e = 0$; $f = 1$, $g = 1$, ac proinde $S = (24 - 9n + 3n^2)\left(\frac{3}{2}\right)^n - 24$.

III. Sint duæ series, altera quadratorum, sive algebrica secundi ordinis $1, 4, 9, 16, 25$, &c. altera geometrica $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}$, &c. singulisque terminis invicem multiplicatis, exurgat nova series $\frac{1}{3}, \frac{4}{9}, \frac{9}{27}, \frac{16}{81}, \frac{25}{243}$, &c. Quoniam primæ seriei terminus generalis $= n^2$, secundæ $= \frac{1}{3^n}$, tertiæ seriei erit $= n^2 \cdot \frac{1}{3^n}$. Ergo ob $m = \frac{1}{3}$ sumpta secunda formula generali, factisque solitis comparatio-

tionibus, invenitur $m = \frac{1}{3}$; $e = 0$; $f = 0$; $g = 1$;

proindeque $S = \frac{3}{2} - \frac{1}{3^n}\left(\frac{3}{2} + \frac{3}{2}n + \frac{1}{2}n^2\right)$. Quod si series in infinitum excurrat, fiet $S = \frac{3}{2}$.

746. *Coroll.* 3. Si serierum, de quibus est quaestio, terminus generalis fuerit $(e + fn + gn^2 + hn^3)m^n$, summa ipsi respondens erit $(A + Bn + Cn^2 + Dn^3)m^n - A$; quamobrem operando ut supra, quantitatum A, B, C, D valores definientur; & sic de seriebus terminos, & summas generales altiores habentibus.

Idem aliter.

Eadem methodo retenta, quoniam tam serierum algebricarum cujuscumque ordinis, quam serierum geometricarum primi ordinis termini generales sunt dati, erit $(A(m-1) + B + (m-1)Bn)m^{n-1} = (a + \overline{n-1} \times b)pm^{n-1}$, seu $A(m-1) + B + (m-1)Bn = ap - bp + bpn$. Hinc $Bn(m-1) = bpn$, & $B = \frac{bp}{m-1}$; $A(m-1) + B = ap - bp$, & $A = \frac{ap - bp}{m-1} - \frac{bp}{(m-1)^2} =$

ap

$$\frac{ap(m-1)-bp(m-1)-bp}{(m-1)^2}=\frac{ap(m-1)-bmp}{(m-1)^2};$$

quare ſubſtituendo hujuſmodi valores in $S=$

$(A+Bn)m^n-A$, provenit $S=$

$\left(\frac{ap(m-1)-bmp}{(m-1)^2}+\frac{bpn}{m-1}\right)m^n-$

$\frac{ap(m-1)+bmp}{(m-1)^2}$, ubi a deſignat primum ſeriei algebricæ terminum, b primam ejuſdem differentiam, p primum ſeriei geometricæ terminum, m denominatorem rationis. Sic ſeriei arithmeticæ 2, 5, 8, 11, 14, &c. nec non ſeriei geometricæ 1, 4, 16, 64, 256, &c. ſingulis terminis invicem ductis, nova emergit ſeries 2, 20, 128, 704, 3584, &c. in qua ob $a=2$, $b=3$, $p=1$, $m=4$, terminus generalis T erit

$=(2+3(n-1))4^{n-1}$; & ſumma $S=$
$\left(-\frac{2}{3}+n\right)4^n+\frac{2}{3}$.

Rurſus ſi ex ſerierum 1, 3, 5, 7, 9, &c. termi-
$\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \frac{1}{48}$, &c.
nis ſingulis invicem ductis nova formetur ſeries $\frac{1}{3}, \frac{3}{6}, \frac{5}{12}, \frac{7}{24}, \frac{9}{48}$, &c. erit $a=1$, $b=2$, p
$=$

$= \frac{1}{3}$, $m = \frac{1}{2}$ quamobrem provenit $T =$

$\left(1 + 2(n-1)\frac{1}{3}\right)\left(\frac{1}{2}\right)^{n-1}$; $S = (-2 - \frac{4}{3}n)\frac{1}{2^n}$
$+ 2.$

Similiter ſi determinanda ſit ſumma generalis $(A + Bn + Cn^2)m^n - A$, quoniam datur ejus terminus generalis $T = (a + \frac{(n-1)}{1}b + \frac{(n-1)(n-2)}{1 \cdot 2}c)pm^{n-1}$, ſumpto alio ipſius ſummæ generalis termino generali nuper expoſito, factaque ejuſdem cum hoc T (quem ſic ſcribo, $T = ap - bp + cp + bpn - \frac{3cpn}{2} + cpn^2$) ſolita comparatione, invenio $C = \frac{cp}{2(m-1)}$; $B(m-1) + 2C = bp - \frac{3cp}{2}$, unde $B(m-1) = \frac{2bp - 3cp}{2} - \frac{2cp}{2(m-1)} = \frac{2bp(m-1) - 3cp(m-1) - 2cp}{2(m-1)}$, & $B = \frac{2bp(m-1) - cp(3m-1)}{2(m-1)^2}$.

A

$$A(m-1) = ap - bp + cp - \frac{2bp(m-1) + cp(3m-1) + cp(m-1)}{2(m-1)^2} =$$

$$\left.\begin{array}{l} 2ap(m-1)^2 - 2bp(m-1)^2 + 2cp(m-1)^2 \\ \quad - 2bp(m-1) + cp(m-1) \\ \quad + cp(3m-1) \end{array}\right\} : 2(m-1)^2, \text{ quare}$$

$$A = \frac{2ap(m-1)^2 - 2bp(m-1) + 2cpm^2}{2(m-1)^3}; \text{ ac tandem}$$

$$S = \left\{ \begin{array}{l} \dfrac{2ap(m-1)^2 - 2bp(m-1) + 2cpm^2}{2(m-1)^3} + \\ \dfrac{2bp(m-1) - cp(3m-1))n}{2(m-1)^2} + \dfrac{cp}{2(m-1)} n^2 \end{array} \right\} m^n$$

$$- \frac{2ap(m-1)^2 + 2bpm(m-1) - 2cpm^2}{2(m-1)^3}$$

ubi a exprimit primum feriei algebricæ terminum, b primam differentiam, c fecundam, p primum feriei geometricæ terminum, m denominatorem rationis.

Sic fi rurfus feries algebrica fecundi ordinis $\frac{3}{2}, \frac{9}{2}, \frac{18}{2}, \frac{30}{2}$, &c. multiplicetur per feriem geometricam $2, 3, \frac{9}{2}, \frac{27}{4}, \frac{81}{8}$, &c. ita ut nova confurgat fe-

feries $\frac{3 \cdot 2}{2}$, $\frac{9 \cdot 3}{2}$, $\frac{18 \cdot 9}{2 \cdot 2}$, $\frac{30 \cdot 27}{2 \cdot 4}$, $\frac{45 \cdot 81}{2 \cdot 8}$, &c.

ftabit $a = \frac{3}{2}$, $b = \frac{6}{2}$, $c = \frac{3}{2}$, $p = 2$, $m = \frac{3}{2}$.

Itaque reperietur $T = (n + n^2)\left(\frac{3}{2}\right)^n$; $S =$

$(24 - 9n + 3n^2)\left(\frac{3}{2}\right)^n - 24$; prorfus ut fupra.

Hac fimplici methodo; datis feriebus algebrico-geometricis altiorum ordinum, termini ac fummæ generales haberi poterunt.

747. *Scholion.* Me non monente, palam eft, dato feriei algebrico-geometricæ termino generali

$(e + fn + gn^2 + hn^3 + \&c.)\, m^n$, feriem ad ipfum pertinentem ex feriebus componentibus derivari poffe. Quum enim loco e, f, g, h, &c. dati fint numeri, ponatur fucceffive $n = 1, 2, 3, 4$, &c. & finguli numeri hinc prodeuntes difponantur in feriem, quæ erit algebrica illius ordinis, quem indicat maximus exponens ipfius n. Deinde numero expreffo per m formetur alia feries, quæ erit geometrica primi ordinis, ftatuendo fucceffive ut ante loco n numeros 1, 2, 3, 4, &c. Tum finguli harum ferierum refpondentes termini inter fe multiplicentur, & nova feries exinde orta erit feries quæfita.

748.

748. Propositio VII. *Series algebrico-geometricæ cujuscumque ordinis sub hac forma* $(A + Bn + Cn^2 + Dn^3 + \&c.)K^n$ *sunt series recurrentes ejusdem ordinis.*

Dem. Vidimus AK^n esse seriem recurrentem primi ordinis, quod etiam per se patet.

Seriei algebrico-geometricæ secundi ordinis terminus generalis est $(A + Bn)K^n$, ex quo formatur series $(A + B)K$, $(A + 2B)K^2$, $(A + 3B)K^3$, &c. ponendo successive $1, 2, 3, 4$, &c. pro n. Jam hæc series nascitur, si duo termini antecedentes multiplicentur, ultimus per $2K$, penultimus per $-K^2$; quod fit manifestum, ducendo primo $(A + (n-1)B)K^{n-1}$ in $2K$, deinde $(A + (n-2)B)K^{n-2}$ in $-K^2$, productorum enim summa $= (A + Bn)K^n$; quare liquet, seriem istam esse recurrentem secundi ordinis.

Series algebrico geometrica tertii ordinis terminum habet generalem $(A + Bn + Cn^2)K^n$, quapropter hæc series emergit, si tres termini antecedentes (initio facto ab ultimo) multiplicentur, hic ultimus per $3K$, penultimus per $-3K^2$, antepenultimus per K^3, adeoque series ista est recurrens tertii ordinis.

Series pariter algebrico-geometrica quarti ordinis,

cujus terminus generalis $(A + Bn + Cn^2 + Dn^3) K^n$, est recurrens quarti ordinis; provenit enim, si quatuor termini antecedentes, incipiendo de more ab ultimo, multiplicentur, ultimus per $4K$, penultimus per $-6K^2$, antepenultimus per $4K^3$, & qui hunc præit, per $-K^4$.

Generatim quum dicti multiplicatores, ordine superiori retento, sint $+K$; $+2K, -K^2$; $+3K, -3K^2, +K^3$; $+4K, -6K^2, +4K^3, -K^4$; &c. facto terminorum in formula

$$(A + Bn + Cn^2 + Dn^3 + \&c.) K^n$$

existentium numero $= m$, si multiplicentur termini antecedentes, ultimus per mK, penultimus per

$$-\frac{m(m-1)}{2} K^2, \text{ antepenultimus per}$$

$$\frac{m(m-1)(m-2)}{2 \cdot 3} K^3, \text{ \& qui hunc præit, per}$$

$$-\frac{m(m-1)(m-2)(m-3)}{2 \cdot 3 \cdot 4} K^4, \text{ atque ita porro,}$$

seriei algebrico-geometricæ terminus generalis per dictam formulam designatus, seriem recurrentem ejusdem ordinis m exhibebit. *Quod erat* &c.

749 *Coroll.* 1. Si formetur æquatio $x^m - mKx^{m-1}$

$$-\frac{m(m-1)}{2} K^2 x^{m-2} - \frac{m(m-1)(m-2)}{2 \cdot 3} K^3 x^{m-3}$$

$+$

$+$ &c. $= 0$, tot in hac aderunt radices ipsi K æquales, quot unitates habet exponens m; si enim ex. gr. fiat $x = 3$, æquatio evadit $x^3 - 3Kx^2 - 3K^2x - K^3 = 0$, ubi tres continentur radices singulæ $= K$. Quamobrem si quantitates, quæ multiplicare debent terminos antecedentes jam memoratos, incipiendo ab ultimo, sint t, s, r, q, &c. erit $mK = t$; $-$

$$\frac{m(m-1)}{2}K^2 = s; \frac{m(m-1)(m-2)}{2 \cdot 3}K^3 = r; -$$

$$\frac{m(m-1)(m-2)(m-3)}{2 \cdot 3 \cdot 4}K^4 = q;$$ &c. quibus valoribus in nuper adsumpta æquatione cum signorum mutatione suffectis, emergit $x^m - tx^{m-1} + sx^{m-2} - rx^{m-3} + qx^{m-4} -$ &c. $= 0$.

750. *Coroll.* 2. Quotiescumque igitur æquatio sic genita radicibus omnibus invicem æqualibus constet, terminus generalis semper expositam formam

$[A + Bn + Cn^2 + Dn^3 +$ &c.$] K^n$ retinebit, & una ex dictis radicibus æqualibus stabit pro K.

751. Propositio VIII. *Datis nuper expositis seriebus recurrentibus, cognitaque singularum lege, tam terminum, quam summam generalem pro earum aggregatis investigare.*

Resolutio. Quoniam propositæ seriei recurrentis, cujus terminus generalis quæritur, datur lex, terminorum antecedentium memoratorum ab ultimo

incipientium multiplicatores sint φ, π, μ, &c. quorum numerus r. Ex iis conflata æquatio $x^r - \varphi x^{r-1} + \pi x^{r-2} - \mu x^{r-3} +$ &c. totidem continebit radices, puta a, b, c, &c. In hac æquatione tres casus, qui dari possunt, contemplor; etenim vel omnes ibi adsunt radices inæquales, vel omnes æquales, vel partim æquales, partim inæquales. Primus casus respicit æquationem $x^n - \alpha x^{n-1} + \beta x^{n-2} - \gamma x^{n-3}$ + &c. adeoque terminus generalis respondens propositæ seriei tunc erit $AK^n + BH^n + CI^n +$ &c. Secundus casus pertinet ad æquationem $x^m - tx^{m-1} + sx^{m-2} - rx^{m-3} +$ &c. $= o$; quare terminus generalis ipsi respondens tunc erit $[A + Bn + Cn^2 + \&c.]\, K^n$. Tertius casus agnoscit nuperam æquationem $x^r - \varphi x^{r-1} + \pi x^{r-2} - \mu x^{r-3} +$ &c. $= o$, ac proinde terminus generalis huic seriei respondens tunc erit duorum antecedentium terminorum generalium aggregatum, nempe $AK^n + BH^n + CI^n +$ &c. $+ [D + En + Fn^2 + \&c.]\, N^n$. Quum igitur ex æquatio-

tionis $x^r - \varphi x^{r-1} + \pi x^{r-2} - \mu x^{r-3} + \&c. = 0$ pro re nata formandæ radicibus habeantur ſingillatim valores quantitatum K, H, I, N, &c. queis formula termini generalis conſtat; ut quantitates quoque A, B, C, D, &c. innoteſcant, terminus generalis, ubi ipſarum K, H, I, N, &c. valores ſint ſubſtituti, comparetur cum primis ſeriei propoſitæ terminis, quorum numerus ſit r, ſtatuendo prius ſucceſſive pro n numeros 1, 2, 3, 4, &c. deinde per idoneas atque iteratas comparationes valores ipſarum A, B, C, D, &c. detegantur. Tandem tam harum, quam quantitatum K, H, I, N, &c. valoribus repertis in aſſumpto termino generali ſuffectis, terminus iſte quæſitus, & cum eo ſumma generalis obtinebitur. *Quod erat* &c.

Calculi progreſſum ſequens illuſtrabit.

Exemplum. Inveniri debeant tam terminus, quam ſumma generalis ſeriei

0, 1, 1, 3, 29, 179, 837, &c. . .

in qua quilibet terminus oritur ex quatuor præcedentium aggregato, hac conditione, ut primus qui hoſce ſtatim ſubſequitur, multiplicetur per 8, ſecundus per -23, tertius per 28, quartus per -12. Quum autem quatuor ſint multiplicatores, per quos ſeriei lex innoteſcit, quatuor adſunt coëfficientes ſigno contrario afficiendi pro æquatione quatuor graduum formanda, cujus radices indeterminatarum K, H, &c. valoribus obtinendis inſerviant. Hæc igitur æquatio erit $x^4 - 8x^3 + 23x^2 - 28x + 12 = 0$, in qua dicti multiplicatores locum tenent ipſis conve-

nientem. Sed hujus æquationis radices sunt 1, 2, 2, 3, duæ nimirum æquales, ac duæ inæquales; ergo terminus generalis ad seriem propositam attinens

exprimetur per $AK^n + BH^n + (E + Fn) M^n$, in qua ipsæ K, H, M per dictas æquationis radices determinantur, ac proinde fiet $K = 1$, $H = 3$, M

$= 2$, adeoque terminus generalis evadet $A . 1^n +$

$B . 3^n + [E + Fn] 2^n$. Pro singulis ipsarum A, B, E, F valoribus determinandis in hoc termino generali ponatur $n = 1$, & instituatur æquatio cum primo seriei propositæ termino 0; deinde fiat $n = 2$, ac formetur alia æquatio cum secundo ejusdem seriei termino 1, & sic deinceps usque ad quartum terminum; ex quo prodit

$$(P) \dots \left\{ \begin{array}{l} A + 3B + 2E + 2F = 0 \\ A + 9B + 4E + 8F = 1 \\ A + 27B + 8E + 24F = 1 \\ A + 81B + 16E + 64F = 3 \end{array} \right.$$

Si a secunda æquatione primam demas, a tertia secundam, & a quarta tertiam, erit

$$(Q) \dots \left\{ \begin{array}{l} 6B + 2E + 6F = 1 \\ 18B + 4E + 16F = 0 \\ 54B + 8E + 40F = 2 \end{array} \right.$$

A secunda subtrahe primam ductam in 3, & a tertia secundam itidem per 3 multiplicatam, & habebis

(R)

$$(R) \ldots\ldots \begin{cases} -2E-2F=-3 \\ -4E-8F=2 \end{cases}$$

A secunda subduc primam ductam in 2, ac fiet $-4F=8$, seu $F=-2$; quo valore in alterutra æquationum (R) suffecto, stabit $E=\frac{7}{2}$. In aliqua ex tribus æquationibus (Q) subrogentur valores ipsarum E, F, & emerget $B=1$. Tandem in aliqua ex quatuor æquationibus (P) substituantur valores ipsarum B, E, F, eritque $A=-6$. Omnium A, B, E, F valores inventi collocentur in formula $A.1^n+B.3^n+[E+Fn]2^n$; proveniet terminus generalis $-6.1^n+1.3^n+\left(\frac{7}{2}-2n\right)2^n$, cui respondet summa $-6n+\frac{3[3^n-3]}{2}+[11-4n]2^n-11$; est enim termini -6.1^n summa $=-6n$; termini 1.3^n summa $=\frac{3[3^n-3]}{2}$; ac termini $\left[\frac{7}{2}-2n\right]2^n$ summa $=[11-4n]2^n-11$.

Qui plura, & præclara per methodos consimiles cupit, adeat Vincentii Riccati *Commentarium de seriebus*, & Gherlii *Elem. di matem. pure* T. 2. Inter.

terea tres propositiones sequentes, quamvis methodos expositas non continuent, huc in Tironum gratiam transferre, haud inutile censeo, videlicet.

752. Propositio IX. *Formulam generalem pro obtinenda summa omnium serierum fractarum, quarum numeratores progressionem servant arithmeticam, denominatores geometricam, exbibere.*

Resolutio. Hujusmodi series generalis $\frac{a}{b}$, $\frac{a+d}{bq}$, $\frac{a+2d}{bq^2}$, $\frac{a+3d}{bq^3}$, &c. collocentur sub hac forma

$\frac{a}{b}$; $\frac{a}{bq}+\frac{d}{bq}$; $\frac{a}{bq^2}+\frac{d}{bq^2}+\frac{d}{bq^2}$; $\frac{a}{bq^3}+\frac{d}{bq^3}+\frac{d}{bq^3}+\frac{d}{bq^3}$; &c. separentur series progressionem geometricam efformantes, nempe

$\div \frac{a}{b}, \frac{a}{bq}, \frac{a}{bq^2}, \frac{a}{bq^3}$, &c. cujus summa $=\frac{aq}{bq-b}$;

$\div \frac{d}{bq}, \frac{d}{bq^2}, \frac{d}{bq^3}$, &c. cujus summa $=\frac{d}{bq-b}$.

$\div \frac{d}{bq^2}, \frac{d}{bq^3}$, &c. ...... cujus summa $=\frac{d}{bq^2-bq}$.

$\div \frac{d}{bq^3}$, &c. .... cujus summa $=\frac{d}{bq^3-bq^2}$;

&c.

Ex

Ex hisce summis (prima excepta) consurgit series $\frac{d}{bq-b}$, $\frac{d}{bq^2-bq}$, $\frac{d}{bq^3-bq^2}$, &c. cujus summa $=\frac{dq}{bq^2-2bq+b}$; quare si huic addatur prima $\frac{aq}{bq-b}$, nascetur $\frac{aq^2-aq+dq}{bq^2-2bq+b}$ proferens summam totius seriei propositæ. *Quod erat* &c.

753. Propositio X. *Seriei* $1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\frac{1}{5^{2n}}$ &c. *summam indagare.*

Resolutio. Esto $y = \sin x$, seu $y = x - \frac{x^3}{2\cdot 3} + \frac{x^5}{2\cdot 3\cdot 4\cdot 5} - \frac{x^7}{2\cdot 3\cdots 7} +$ &c. Statuatur $y = 0$, erit $x - \frac{x^3}{2\cdot 3} + \frac{x^5}{2\cdot 3\cdot 4\cdot 5} - \frac{x^7}{2\cdot 3\cdots 7} +$ &c. $= 0$; dividendo per x, fit $1 - \frac{x^2}{2\cdot 3} + \frac{x^4}{2\cdot 3\cdot 4\cdot 5} - \frac{x^6}{2\cdot 3\cdots 7} +$ &c. $= 0$; cujus æquationis, ob termi-

minos infinitos, radices erunt pariter infinitæ. At posita circuli ſemicircumferentia $=\pi$, quum arcus ſinui $=0$ reſpondentes ſint π, 2π, 3π, 4π, &c. (215.), arcus illi erunt hujus poſtremæ æquationis totidem radices, nam ob factam diviſionem per x, primus arcus $=0$ ſinui $=0$ reſpondens evanuit. Fiat nunc

$$x=\frac{1}{z};\ \text{erit}\ 1-\frac{1}{2\cdot3\cdot z^2}+\frac{1}{2\cdot3\cdot4\cdot5\cdot z^4}-\frac{1}{2\cdot3\cdots\cdot7\cdot z^6}+\&c.=0.$$

Ut autem iſta æquatio in aliam, quæ a maxima incognitæ potentia incipiat, commutetur, finge, ipſam eſſe completam, nempe (ob incognitæ dimenſiones ſemper pares, & quantitati $2n$ æquales) fieri

$$1-\frac{1}{2\cdot3\cdot z^2}+\frac{1}{2\cdot3\cdot4\cdot5\cdot z^4}-\frac{1}{2\cdot3\cdots\cdots7\cdot z^6}+\cdots\cdot\pm\frac{1}{2\cdot3\cdot4\cdots(2n+1)\,z^{n}}=0,$$

quæ (per z^{2n} multiplicando) evadet

$$z^{2n}-\frac{z^{2n-2}}{2\cdot3}+\frac{z^{2n-4}}{2\cdot3\cdot4\cdot5}-\frac{z^{2n-6}}{2\cdot3\cdots7}+\cdots\cdot\pm\frac{1}{2\cdot3\cdot4\cdots\cdot 2n+1}=0.$$

Hujus æquationis terminos dimenſionibus paribus affectos habentis radices $\left(\text{ob } zz=\frac{1}{xx}\right)$ ſunt $\frac{1}{\pi\pi}$,

$\frac{1}{4\pi\pi}$, $\frac{1}{9\pi\pi}$, $\frac{1}{16\pi\pi}$, &c. quamobrem quoniam secundi termini coëfficiens, ſigno mutato, exprimit radicum ipſarum ſummam, habebitur $\frac{1}{2 \cdot 3}$, ſeu $\frac{1}{6}$

$= \frac{1}{\pi\pi} + \frac{1}{4\pi\pi} + \frac{1}{9\pi\pi} + \frac{1}{16\pi\pi} +$ &c. adeoque

$\frac{1}{6}\pi\pi = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} +$ &c. $= 1 + \frac{1}{2^2} + \frac{1}{3^2}$

$+ \frac{1}{4^2} +$ &c. Jam vero radices $\frac{1}{\pi\pi}$, $\frac{1}{4\pi\pi}$, $\frac{1}{9\pi\pi}$,

$\frac{1}{16\pi\pi}$, &c. pro ſuorum ſingulorum quadratorum ſumma habent $\frac{1}{\pi^4} + \frac{1}{4^2\pi^4} + \frac{1}{9^2\pi^4} + \frac{1}{16^2\pi^4} +$ &c. quæ ſumma ex dictis (434.) adæquat

$\left(\left(\frac{1}{2 \cdot 3} \right)^2 - \frac{2}{2 \cdot 3 \cdot 4 \cdot 5} \right) \pi^4$, ſeu $\left(\frac{1}{36} - \frac{1}{60} \right) \pi^4$

$= \frac{1}{90}\pi^4 = 1 + \frac{1}{4^2} + \frac{1}{9^2} + \frac{1}{16^2} +$ &c. $= 1 +$

$\frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} +$ &c. Itidem pro ſumma cuborum erit

erit $\left(\left(\frac{1}{2 \cdot 3}\right) 90 - \left(\frac{1}{2 \cdot 3 \cdot 4 \cdot 5}\right)\left(\frac{1}{2 \cdot 3}\right) + \frac{3}{2 \cdot 3 \cdots\cdots 7} \right) \pi^6 = \frac{\pi^6}{940} = 1 + \frac{1}{4^3} + \frac{1}{9^3} + \frac{1}{16^3} +$ &c. $= 1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} +$ &c.

Similiter agendo, obtinebitur summa seriei $1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} +$ &c. & sic deinceps. *Quod erat* &c.

754. *Coroll.* 1. Si fuerit $P = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} +$ &c. multiplicetur hæc series per $\frac{1}{2^{2n}}$; fiet $\frac{1}{2^{2n}} P = \frac{1}{2^{2n}} + \frac{1}{4^{2n}} + \frac{1}{6^{2n}} + \frac{1}{8^{2n}} +$ &c.

Subtracta hac secunda serie a prima, residuum erit

$$\left(1 - \frac{1}{2^{2n}}\right) P = 1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{9^{2n}} + \text{\&c.}$$

755.

755. *Coroll.* 2. Multiplicata per 2 hac tertia serie, dematur ab ipsa series prima, residuum emerget $[1-2^{1-2n}]P = 1 - \frac{1}{2^{2n}} + \frac{1}{3^{2n}} - \frac{1}{4^{2n}} + \frac{1}{5^{2n}} - \frac{1}{6^{2n}} +$ &c.

Quare determinato prius numero n, haberi potest summa serierum

$$1 \pm \frac{1}{2^{2n}} + \frac{1}{3^{2n}} \pm \frac{1}{4^{2n}} + \frac{1}{5^{2n}} \pm \frac{1}{6^{2n}} + \text{\&c.}$$

$$1 + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{9^{2n}} + \text{\&c.}$$

756. Propositio XI. *Esto series* A, *in qua terminorum numeratores sint constantes, & denominatores numeri naturales elevati ad quamlibet potestatem* n, *positivam aut negativam, integram aut fractam.*

$$A\ldots \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \frac{1}{7^n} + \frac{1}{8^n} + \text{\&c.}$$

$$B\ldots \frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} + \frac{1}{10^n} + \text{\&c.}$$

$$C\ldots \frac{1}{3^n} + \frac{1}{6^n} + \frac{1}{9^n} + \frac{1}{12^n} + \text{\&c.}$$

D....

$D \ldots \frac{1}{4^n} + \frac{1}{8^n} + \frac{1}{12^n} + \frac{1}{16^n} +$ &c.

&c.

Series autem B *constet ex seriei* A *terminis captis in locis paribus.*

Series C *formetur ex seriei* A *terminis post duos numeros successive desumptis.*

Series D *componatur ex ejusdem* A *terminis post tres numeros gradatim acceptis; Et sic quousque libeat; Invenienda est relatio inter seriem* A, *& singulas series* B, C, D, &c. *cummodo terminorum numerus in seriebus comparatis idem sit idque in infinitum.*

Resolutio. Seriem *B* considerans, statim video, singulos ejus terminos esse partem $\frac{1}{2^n}$ singulorum terminorum seriei *A*; est enim $\frac{1}{1^n} : \frac{1}{2^n} = 2^n$; $\frac{1}{2^n} : \frac{1}{4^n} = \frac{4^n}{2^n} = 2^n$, & sic de reliquis; adeoque series *A* erit ad seriem *B*, ut 2^n ad 1.

Similiter si seriem *C* contemplor, percipio, singulos ejus terminos esse partem $\frac{1}{3^n}$ seriei *A*; siquidem $\frac{1}{1^n} : \frac{1}{3^n} = 3^n$, & $\frac{1}{2^n} : \frac{1}{6^n} = \frac{6^n}{3^n} = 3^n$

qua-

quapropter series *A* erit ad seriem *C*, ut 3^n ad 1.

Pari facilitate ostendam, esse $A : D = 4^n : 1$, atque ita porro. Relationes igitur inter *A*, & singulas *B*, *C*, *D*, &c. habentur per serierum comparatarum denominatores in primis earum terminis existentes ad unitatem relatos. *Quod erat* &c.

757. *Coroll.* Alias quoque comparationes instituere possumus; ex. gr. inter seriem terminorum imparium $\frac{1}{1^n} + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} +$ &c. Et seriem terminorum parium $\frac{1}{2^n} + \frac{1}{4^n} + \frac{1}{6^n} + \frac{1}{8^n} +$ &c. Quum enim series ex duabus hisce seriebus conflata sit ad secundam, ut 2^n ad 1, liquet, seriem terminorum imparium esse ad seriem parium, ut $2^n - 1$ ad 1. Itidem series *C* est ad seriem *D*, ut $\frac{1}{3^n}$ ad $\frac{1}{4^n}$, sive ut 4^n ad 3^n; &c.

CAPUT V.

DE FRACTIONIBUS CONTINUIS.

758. Definitio. *Fractio continua* est illa, cujus denominator duabus constat partibus, una integra, & altera fracta; hujus autem fractionis deno-

nominator est partim integer, partim fractus, atque ita porro; talis est $\alpha + \cfrac{b}{\beta + \cfrac{c}{\gamma + \cfrac{d}{\delta + \&c.}}}$

759. Propositio I. *Fractionis continuae ortum exponere.*

Resolutio. Esto fractio impropria, seu spuria $\frac{a}{b}$, in qua nempe $a > b$. Facta divisione, quotus sit α, & residuum c, seu $\frac{a}{b} = \alpha + \frac{c}{b}$. Quoniam $b > c$, invertatur fractio, ac dividatur b per c, sitque quotus $\frac{b}{c} = \beta + \frac{d}{c}$. Quia vero $c > d$, ponatur $\frac{c}{d} = \gamma + \frac{e}{d}$. Similiter ob $d > e$, fiat $\frac{d}{e} = \delta + \frac{f}{e}$, & sic deinceps, quae operatio congruit cum regula [29.] quam dedimus pro maximo communi divisore inveniendo. Habemus igitur [1.°] $\frac{a}{b} = \alpha + \frac{c}{b}$; [2.°] $\frac{b}{c} = \beta + \frac{d}{c}$; [3.°] $\frac{c}{d} = \gamma + \frac{e}{d}$; [4.°] $\frac{d}{e} = \delta + \frac{f}{e}$, &c.

Ex secunda fractione [invertendo aequationum membra]

bra] fit $\frac{c}{b} = \frac{1}{\beta + \frac{d}{c}} = A$. Ex tertia $\frac{d}{c} = \frac{1}{\gamma + \frac{e}{d}}$ $= B$. Ex quarta $\frac{e}{d} = \frac{1}{\delta + \frac{f}{e}} = C$; &c.

Subſtitue valorem A in prima fractione pro $\frac{c}{b}$, & in fractione A pro $\frac{d}{c}$ ſuffice ſuum valorem B; pari modo in fractione B ſubrogando valorem C pro $\frac{e}{d}$, &c. tandem proveniet $\frac{a}{b} = \alpha + \cfrac{1}{\beta + \cfrac{1}{\gamma + \cfrac{1}{\delta + \&c.}}}$

Quod erat &c.

Exemplum. Sit fractio communis $\frac{3347}{2241}$ in fractionem continuam redigenda. Diviſionibus ſucceſſivis peractis, prodit

$$\frac{3347}{2241} = 1 + \frac{1106}{2241}$$

$$\frac{2241}{1106} = 2 + \frac{29}{1106}$$

$$\frac{1106}{29} = 38 + \frac{4}{29}$$

$$\frac{29}{4} = 7 + \frac{1}{4}$$

$$\frac{4}{1} = 4$$

ubi apparet, numeros integros in calculi typo existentes, primo excepto, esse fractionis continuæ quæsitæ denominatores pro β, γ, δ, &c. substituendos;

quare $\frac{3347}{2241} = 1 + \cfrac{1}{2 + \cfrac{1}{38 + \cfrac{1}{7 + \cfrac{1}{4}}}}$

760. *Coroll.* 1. Omnes denominatores sunt igitur numeri integri rationales; numeratores totidem unitates.

761. *Coroll.* 2. Si fractio $\frac{a}{b}$ fuisset fractio propria, seu vera, oportebat prius denominatorem per nume-

meratorem dividere. Sic $\frac{14}{75} = \cfrac{1}{5+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{4}}}}$; nam

$\frac{75}{14} = 5 + \frac{5}{14}$, quare pro 5 ſcribi debet $\frac{1}{5}$. Hinc

762\. *Coroll.* 3. Si fractio communis propoſita ſit vera, primus fractionis continuæ terminus erit ſemper fractio vera, & contra ſemper quantitas integra, ſi fractio data ſit ſpuria.

763\. *Coroll.* 4. Si $\frac{a}{b}$ ſit quantitas rationalis, operatio in ejuſdem reductione ad fractionem continuam terminabit, quia in ſucceſſivis divisionibus tandem devenietur ad numerum, qui dividi debeat per unitatem, ut etiam allatum Exemplum oſtendit. Sed ſi $\frac{a}{b}$ ſit quantitas irrationalis, operatio nunquam terminari poterit; quantitas enim irrationalis non niſi per ſeriem infinitam rationaliter exprimi poteſt.

764\. Propoſitio II. *Fractionem continuam ad fractionem communem reducere.*

Reſolutio. Duæ fractiones, incipiendo ab ultima, ad eundem denominatorem operatione retrograda

 ſuc-

successive reducantur; summa erit fractio communis quæsita.

Exemplum. Esto fractio continua

$$\alpha + \cfrac{b}{\beta + \cfrac{c}{\gamma + \cfrac{d}{\delta + \cfrac{e}{\varepsilon + \&c.}}}}$$

si duas primas fractiônes $\alpha + \cfrac{b}{\beta + \cfrac{c}{\gamma}}$ reducere libeat, incipio a fractione $\frac{b}{\beta + \frac{c}{\gamma}}$, & obtineo $\frac{b\gamma}{\beta\gamma + c}$, quare $\alpha + \frac{b\gamma}{\beta\gamma + c} = \frac{\alpha\beta\gamma + \alpha c + b\gamma}{\beta\gamma + c}$. Si primi termini $\alpha + \cfrac{b}{\beta + \cfrac{c}{\gamma + \frac{d}{\delta}}}$ fuerint ad fractionem communem reducendi, ordior a fractione $\frac{c}{\gamma + \frac{d}{\delta}}$, & habeo $\frac{c\delta}{\gamma\delta + d}$; deinde idem exequor cum fractione $\cfrac{b}{\beta + \cfrac{c\delta}{\gamma\delta + d}}$, ac prodit $\frac{b\gamma\delta + bd}{\beta\gamma\delta + \beta d + c\delta}$; tum addendo α,

re-

reduco $\alpha + \frac{b\gamma\delta + bd}{\beta\gamma\delta + \beta d + c\delta}$, & reperio

$\frac{\alpha\beta\gamma\delta + \alpha\beta d + \alpha c\delta + b\gamma\delta + bd}{\beta\gamma\delta + \beta d + c\delta}$, & ſic deinceps, ſi libeat calculum protrahere.

Sic ad numerum communem $\frac{3347}{2241}$ redigitur fraſtio continua ſuperior $1 + \cfrac{1}{2 + \cfrac{1}{38 + \cfrac{1}{7 + \cfrac{1}{4}}}}$; nam factis duabus ultimis fractionibus $\frac{1}{7 + \frac{1}{4}} = \frac{4}{29}$, ſuccedunt aliæ duæ fractiones contiguæ, nempe $\frac{1}{38 + \frac{4}{29}} = \frac{29}{1106}$; ſequunturque aliæ duæ proximæ $\frac{1}{2 + \frac{29}{1106}} = \frac{1106}{2241}$, quibus addita unitate, habetur $1 + \frac{1106}{2241} = \frac{3347}{2241}$.

765. *Scholion.* Ex conſideratione terminorum,

 qui-

quibus fractiones istæ communes a fractionibus continuis erutæ componuntur, sequens dimanat

Regula. Statuantur duæ fractiones, prima $\frac{1}{0}$, secunda $\frac{\alpha}{1}$; super istis, ac subsequentibus, quæ invenientur, scribantur ex ordine fractionum continuarum denominatores, incipiendo a primo termino, & eodem ordine retento, sub iisdem scribantur numeratores, nimirum

$$\begin{array}{cccccc} & \alpha & \beta & \gamma & \delta & \epsilon \\ \frac{1}{0}, & \frac{\alpha}{1}, & \frac{\alpha\beta+b}{\beta}, & \frac{\alpha\beta\gamma+\alpha c+b\gamma}{\beta\gamma+c}, & \frac{\alpha\beta\gamma\delta+\alpha\beta d+\alpha c\delta+b\gamma\delta+bd}{\beta\gamma\delta+\beta d+c\delta}, & \\ b & c & d & e & f & \&c. \end{array}$$

Rebus sic dispositis, ut fractionis cujuscumque reductæ numerator habeatur, fractionis ex inventis ultimæ numerator ducatur in litteram ipsi superpositam, deinde fractionis penultimæ numerator ducatur in litteram eidem subscriptam; ex duobus hisce productis fiat aggregatum, quod numeratorem quæsitum exhibebit.

Pro denominatore obtinendo, ex denominatore fractionis ultimæ ducto in litteram superpositam, & ex denominatore penultimæ pariter a læva ducto in litteram subscriptam fiat aggregatum, quod denominatorem exprimet expetitum.

Sic quarti termini numerator prodit ex $(\alpha\beta+b)\gamma+\alpha\times c$, denominator ex $\beta\times\gamma+1\times c$.

766. *Coroll.* 1. Itaque si fuerit fractio continua

$\alpha +$

$$\alpha + \cfrac{1}{\beta + \cfrac{1}{\gamma + \cfrac{1}{\delta + \&c.}}}$$ fractiones communes (quæ *æquivalentes* quoque appellari solent) erunt

$$\alpha, \frac{\alpha\beta+1}{\beta}, \frac{(\alpha\beta+1)\gamma+\alpha}{\beta\gamma+1}, \frac{((\alpha\beta+1)\gamma+\alpha)\delta+\alpha\beta+1}{(\beta\gamma+1)\delta+\beta},$$

$$\frac{((\alpha\beta+1)\gamma\delta+(\beta+\delta)\alpha+1)\varepsilon+(\beta+1)\alpha+\gamma}{((\beta\gamma+1)\delta+\beta)\varepsilon+\beta\gamma+1} \&c.$$

Hinc fractionis continuæ $\frac{3347}{2241}$ fractiones æquivalentes sunt $1, \frac{3}{2}, \frac{115}{77}, \frac{808}{541}, \frac{3347}{2241}$; nam $b = 1$, $c = :$, $d = 1$, $e = 1$; $\alpha = 1$, $\beta = 2$, $\gamma = 38$, $\delta = 7$, $\varepsilon = 4$.

767. *Coroll.* 2. Ergo ad verum fractionis continuæ valorem obtinendum ultima fractio æquivalens est accipienda, præcedentes enim nonnisi proximum exhibent valorem, qui eo magis fit exactus, quo plus ad ultimam appropinquat.

768. *Coroll.* 3. Istæ igitur fractiones sunt continue convergentes versus verum fractionis continuæ valorem.

769. Propositio III. *Fractiones communes, seu æquivalentes, aliter, & quidem commodius exprimere.*

Resolutio. Statue

$$\begin{array}{ll} p = \alpha & p' = 1 \\ q = \beta\alpha + 1 & q' = \beta p' \\ r = \gamma q + p & r' = \gamma q' + p' \\ s = \delta r + q & s' = \delta r' + q' \\ \&c. & \&c. \end{array}$$

ha-

habebis $\alpha = \frac{p}{p'}$; $\frac{\alpha\beta + 1}{\beta} = \frac{q}{q'}$; $\frac{(\alpha\beta + 1)\gamma + \alpha}{\beta\gamma + 1} =$

$\frac{q\gamma + p}{q'\gamma + p'} = \frac{r}{r'}$; &c. ac proinde exurget series

$\frac{p}{p'}$, $\frac{q}{q'}$, $\frac{r}{r'}$, $\frac{s}{s'}$, &c. *Quod erat* &c.

770. *Coroll.* 1. Ergo statim dispicitur, quantitates p, q, r, &c. p', q', r', &c. ejus esse proprietatis, ut fiat $qp' - pq' = 1$; $rq' - qr' = -1$; $s'r - rs' = 1$; &c.

771. *Coroll.* 2. $\frac{q}{q'} - \frac{p}{p'} = \frac{1}{p'q'}$; $\frac{q}{q'} - \frac{r}{r'} = \frac{1}{q'r'}$;

$\frac{s}{s'} - \frac{r}{r'} = \frac{1}{r's'}$; &c. quod indicat, fractiones $\frac{p}{p'}$,

$\frac{q}{q'}$, $\frac{r}{r'}$, &c. differre non posse a vero fractionis $\frac{a}{b}$

valore, nisi quantitate minori quam $\frac{1}{p'q'}$, vel $\frac{1}{q'r'}$,

vel $\frac{1}{r's'}$, &c. ex quo de approximatione ad verum valorem judicium ferre possumus.

772. *Coroll.* 4. Similiter ob $\frac{q}{q'} - \frac{p}{p'} = \frac{1}{p'q'}$; $\frac{r}{r'}$ —

$-\frac{q}{q'} = -\frac{1}{q'r'}; \frac{s}{s'} - \frac{r}{r'} = \frac{1}{r's'}; \frac{t}{t'} - \frac{s}{s'} = -\frac{1}{s't'}$; &c.
facile infertur, differentias inter fractiones æquivalentes se proxime subsequentes seriei $\frac{p}{p'}, \frac{q}{q'}, \frac{r}{r'}$, &c. successive imminui, stante numeratore, & crescente denominatore, adeoque rursus hanc seriem esse convergentem, harumque differentiarum terminos signis diversis alteratim affici.

773. *Coroll.* 5. Præterea habemus $\frac{q'}{p'} = \beta; \frac{r'}{q'} =$

$$\frac{\beta\gamma+1}{\beta} = \gamma + \frac{p'}{q'}; \frac{s'}{r'} = \frac{(\beta\gamma+1)}{\beta\gamma+1}\delta + \frac{\beta}{\beta\gamma+1} =$$

$\delta + \frac{r'}{q'}$; &c.

774. *Coroll.* 3. Quoniam vero $q' > p'$, $r' > q'$, &c. erit $\frac{1}{(p')^2} > \frac{1}{p'q'}; \frac{1}{(q')^2} > \frac{1}{q'r'}$; &c. unde patet, errorem cujuscumque fractionis semper fore minorem unitate divisa per quadratum denominatoris ejusdem fractionis.

775. *Coroll.* 6. Fractiones æquivalentes $\frac{p}{p'}, \frac{q}{q'}, \frac{r}{r'}$, &c. sunt ad minimos terminos reductæ; si enim

ex.

ex. gr. r, & r' communem diviſorem ab unitate diverſum habeant, ſit $r = eg$, $r' = fg$; erit $rq' - qr' = egq' - fgq = g(eq' - fq)$, quod ob $rq' - qr' = -1$ (770.) fieri nequit.

776. *Coroll.* 7. Data fractione quavis, alia inveniri poteſt, qua a data ſubtracta, reſiduum (communi denominatore prætermiſſo) ſit $= \pm 1$; nimirum fractio data redigatur ad fractionem continuam, quæ ad fractionem æquivalentem ultimo termino mutilatam reducta erit fractio quæſita. Sic fractio continua

$$1 + \cfrac{1}{2 + \cfrac{1}{38 + \cfrac{1}{7 + \cfrac{1}{4}}}}$$

reducta ad fractionem æquivalentem fit, ut nuper vidimus, $\frac{3347}{2241}$; ſed ſi omiſſo ultimo termino $\frac{1}{4}$, fractio remanens

$$1 + \cfrac{1}{2 + \cfrac{1}{38 + \cfrac{1}{7}}}$$

reducatur ad fractionem æquivalentem, prodibit $\frac{808}{541}$; quare $\frac{3347}{2241} - \frac{808}{541} = 1810727 - 1810728 = -1$. Hoc enim eſt idem ac ſi omiſſo denominatore, ab $\frac{r}{r'}$ dematur $\frac{q}{q'}$ (771.).

777. Propoſitio IV. *Quantum quælibet fractio* $\frac{p}{p'}$

$\frac{p}{p'}$, *vel* $\frac{q}{q'}$, *&c. ad verum datæ quantitatis* $\frac{a}{b}$ *valorem accedat, indagare.*

Resolutio. Substituendo valores expositos (769.), fiet $\frac{a}{b} = \frac{\alpha\beta+1}{\beta} = \frac{p\beta+1}{p'\beta}$; est autem $\beta = \frac{q-1}{p}$; ergo $\frac{a}{b} = \frac{(\alpha\beta+1)\gamma+\alpha}{\beta\gamma+1} = \frac{(\alpha q-\alpha+p)\gamma+\alpha p}{p(\beta\gamma+1)}$; sed $\alpha = p$, & $\beta\gamma+1 = r'$; ergo $\frac{a}{b} = \frac{q\gamma+p}{q'\gamma+p'}$, & sic deinceps; quapropter habebimus $\frac{a}{b} = \frac{p\beta+1}{p'\beta}$; $\frac{a}{b} = \frac{q\gamma+p}{q'\gamma+p'}$; $\frac{a}{b} = \frac{r\delta+q}{r'\delta+q'}$, $\frac{a}{b} = \frac{s\varepsilon+r}{s'\varepsilon+r'}$; &c. Ut itaque scias quod cupis, quære differentiam ex. gr. inter $\frac{r}{r'}$, & $\frac{a}{b}$, eritque $\frac{a}{b} - \frac{r}{r'} = \frac{r\delta+q}{r'\delta+q'} - \frac{r}{r'} = \frac{qr'-rq'}{r'(r'\delta+q')} = \frac{1}{r'(r'\delta+q')}$ (770.). Quum autem δ adsumatur proxime minor vero valore, puta $\delta < d$, ita ut differentia inter δ & d sit minor unitate, verus valor $r'd+q'$ stabit inter li-

limites $r'\delta+q'$, & $r'(\delta+1)+q'$, nimirum ob $\delta=\frac{s'}{r'}-\frac{q}{r'}$, inter s', & $s'+r'$; Ergo differentia $\frac{a}{b}-\frac{r}{r'}$ latebit inter duos limites $\frac{1}{r's'}$, & $\frac{1}{r'(s'+r')}$, quod indicabit quanta sit approximatio quæsita, & sic de ceteris fractionibus æquivalentibus; *Quod erat* &c.

778. *Coroll.* 1. Hinc erit etiam $\frac{a}{b}=\frac{p}{p'}+\frac{1}{p'\beta}$; $\frac{a}{b}=\frac{q}{q'}+\frac{q\gamma+p}{q'\gamma+p'}-\frac{q}{q'}=\frac{q}{q'}-\frac{pq'-qp'}{q'(q'\gamma+p')}$ $=\frac{q}{q'}-\frac{1}{q'(q'\gamma+p')}$; & ita porro; quamobrem stabit $\frac{a}{b}=\frac{p}{p'}+\frac{1}{p'\beta}$; $\frac{a}{b}=\frac{q}{q'}-\frac{1}{q'(q'\gamma+p')}$; $\frac{a}{b}=\frac{r}{r'}+\frac{1}{r'(r'\delta+q')}$; $\frac{a}{b}=\frac{s}{s'}-\frac{1}{s'(s'\epsilon-r')}$ &c.

Itaque quum valores appropinquantes α, β, γ, &c. sint adsumpti vero minores, numeri isti erunt semper positivi, adeoque veri eorum valores erunt pariter positivi, ut etiam numeri p', q', r', &c. (769.); quapropter differentiæ inter $\frac{a}{b}$, & fractiones $\frac{p}{p'}$, $\frac{q}{q'}$,

r

$\frac{r}{r'}$, &c. erunt per vices negativæ ac poſitivæ, ſeu per vices minores, & majores quantitate $\frac{a}{b}$. Nam

$$\frac{p}{p'} - \frac{a}{b} = -\frac{1}{p'\beta};\ \frac{q}{q'} - \frac{a}{b} = \frac{1}{q'(q'\gamma + p')};\ \frac{r}{r'} - \frac{a}{b} = -\frac{1}{r'(r'\delta + q')};\ \frac{s}{s'} - \frac{a}{b} = \frac{1}{s'(s'\epsilon - r')};\&c.$$

779. *Coroll.* 2. Ergo verus fractionis continuæ valor inter duas quaslibet fractiones æquivalentes ſe ſtatim ſubſequentes conſiſtit, quæ idcirco ſunt ejus duo limites.

780. *Coroll.* 3. Si ſeries $\frac{p}{p'}$, $\frac{q}{q'}$, $\frac{r}{r'}$, &c. dividatur in duas ſeries $\frac{p}{p'}$, $\frac{r}{r'}$, $\frac{t}{t'}$, &c. $\frac{q}{q'}$, $\frac{s}{s'}$, $\frac{u}{u'}$, &c. prima fractiones omnes minores quam $\frac{a}{b}$ comprehendet, quæ idcirco quantitatem $\frac{a}{b}$ verſus ſucceſſive augebuntur; adeoque ſecunda conſtabit fractionibus majoribus quam $\frac{a}{b}$, quæ eandem quantitatem verſus imminuentur.

781.

781. Propositio V. *Differentia inter duas fractiones aequivalentes se statim subsequentes est minima possibilis, ita ut nulla alia fractio inter istas intercedere possit, nisi denominatorem habeat, qui major sit alterutro talium fractionum denominatore.*

Dem. Si negas, duae dictae fractiones sint $\frac{q}{q'}$, $\frac{r}{r'}$, quarum differentia $= \frac{1}{q'r'}$, deturque, si fieri potest, alia fractio $\frac{m}{n}$, cujus valor cadat inter $\frac{q}{q'}$, & $\frac{r}{r'}$, cujusque denominator n minor sit quam q', vel quam r'. Ergo differentia inter $\frac{q}{q'}$, & $\frac{m}{n}$ (quae est $\frac{qn - q'm}{q'n}$, vel $\frac{q'm - qn}{q'n}$) minor foret quam $\frac{1}{q'r'}$; Sed hoc fieri nequit, ut patet; Ergo si $n < r'$, erit $\frac{1}{q'n} > \frac{1}{q'r'}$. Similiter quum differentia inter $\frac{m}{n}$, & $\frac{r}{r'}$ esse nequeat minor quam $\frac{1}{nr'}$, erit major quam $\frac{1}{q'r'}$, si $n < q'$; *Quod erat* &c.

782.

782. *Coroll.* Quum igitur inter duas fractiones, de quibus eſt ſermo, quantitatis datæ $\frac{a}{b}$ valor conſiſtat (779.), quævis fractio æquivalens exprimet quantitatem $\frac{a}{b}$ accuratius quam quæcumque alia fractio, cujus denominator ſit minor denominatore fractionis proxime ſubſequentis. Fractio ex. gr. $\frac{q}{q'}$ exprimet valorem quantitatis $\frac{a}{b}$ accuratius quam quælibet alia fractio $\frac{m}{n}$, in qua ſit $n < r'$.

783. Propoſitio VI. *Caſus inveſtigare, cum inter fractiones æquivalentes novæ fractiones* [*quæ* intermediæ *vocentur*], *proximum quantitatis* $\frac{a}{b}$ *valorem & ipſæ exhibentes, inſeri poſſunt.*

Reſolutio. Seriebus ſuperioribus $\frac{p}{p'}$, $\frac{r}{r'}$, $\frac{t}{t'}$, &c. $\frac{q}{q'}$, $\frac{s}{s'}$, $\frac{u}{u'}$ &c. (780.) ad examen revocatis, ob $p'r - pr' = \gamma p'q + pp' - \gamma pq' - pp'$ (769.) $= \gamma(p'q - pq') = \gamma$ (770.), habemus in prima $\frac{r}{r'} - \frac{p}{p'} = \frac{\gamma}{p'r'}$; $\frac{t}{t'} - \frac{r}{r'}$

=

$= \frac{\epsilon}{r'\,t'}$; &c. In ſecunda vero $\frac{q}{q'} - \frac{s}{s'} = \frac{\delta}{q's'}$; $\frac{s}{s'} - \frac{u}{u'} = \frac{\zeta}{s'u'}$, &c. quapropter ſi numeri ſinguli, γ, δ, ϵ, &c. ſint unitati æquales, redit caſus jam indicâtus (781.). At ſi hujuſmodi numeri ſint unitate majores, inter recenſitas fractiones tot fractiones intermediæ interponi poſſunt, quot unitates ſunt in numeris $\gamma - 1$, $\delta - 1$, $\epsilon - 1$, &c. Ponatur ex. gr. $\gamma = 4$; fiet $r = 4q + p$, & $r' = 4q' + p'$ (769.), quare inter fractiones $\frac{p}{p'}$, & $\frac{r}{r'}$ tres interferri poſſunt fractiones intermediæ $\frac{q+p}{q'+p'}$; $\frac{2q+p}{2q'+p'}$; $\frac{3q+p}{3q'+p'}$.

Idem quoad valores s, & s' intelligendum, ſi fuerit $\delta = 4$, &c. & ita ſemper, quicumque ſit numerus unitate major, quem exprimant γ, δ, ϵ, &c. Etenim ſi fractionum memoratarum ſumantur differentiæ, ob $qp' - pq' = 1$, fiet

$$\frac{q+p}{q'+p'} - \frac{p}{p'} = \frac{1}{p'(q'+p')}$$

$$\frac{2q+p}{2q'+p'} - \frac{q+p}{q'+p'} = \frac{1}{(q'+p')(2q'+p')}$$

$$\frac{3q+p}{3q'+p'} - \frac{2q+p}{2q'+p'} = \frac{1}{(2q'+p')(3q'+p')}$$

r

$$\frac{r}{r'} - \frac{3q+p}{3q'+p'} = \frac{1}{(3q'+p')r'};$$

ubi conſtat, fractiones $\frac{p}{p'}$; $\frac{q+p}{q'+p'}$; $\frac{2q+p}{2q'+p'}$; &c. non ſolum habere denominatores progreſſionem arithmeticam creſcentem ſervantes, ſed ſucceſſive incrementum capere ex $\frac{p}{p'}$ uſque ad $\frac{r}{r'}$, quod earum differentiæ ſunt omnes poſitivæ. Præterea quum iſtæ differentiæ ſint æquales unitati per denominatorum productum diviſæ, valet iterum pro iiſdem quod de aliis conſimilibus demonſtravimus (781.). Accedat, quod quum fractiones recenſitæ ſint minores valore $\frac{a}{b}$, & fractio $\frac{q}{q'}$ ſit eodem valore $\frac{a}{b}$ major (778.), quælibet ex fractionibus, de quibus eſt ſermo, ita fiet quantitati $\frac{a}{b}$ proxima, ut differentia ſit minor quam illa, quæ intercedit inter dictam fractionem, & fractionem $\frac{q}{q'}$. Profecto

$$\frac{p}{p'} - \frac{q}{q'} = \frac{1}{p'q'}$$

$$\frac{q+p}{q'+p'} - \frac{q}{q'} = \frac{1}{(q'+p')q'}$$

$$\frac{2q+p}{2q'+p'} - \frac{q}{q'} = \frac{1}{(2q'+p')q'}$$

$$\frac{3q+p}{3q'+p'} - \frac{q}{q'} = \frac{1}{(3q'+p')q'}$$

$$\frac{r}{r'} - \frac{q}{q'} = \frac{1}{r'q'};$$

quæ differentiæ quum ſint pariter æquales unitati per denominatorum productum diviſæ, dictam demonſtrationem (781.) & ipſæ recipiunt, qua ſtatuitur, nullam fractionem $\frac{m}{n}$ cadere poſſe inter quaſlibet fractiones $\frac{p}{p'}$; $\frac{q+p}{q'+p'}$; $\frac{2q+p}{2q'+p'}$; &c. ſi denominator n ſit minor denominatore alterutrius ex duabus hujuſmodi fractionibus ſe ſtatim ſuccedentibus; ex quo ſequitur, quaslibet ex iſtis fractionibus propius accedere ad quantitatem propoſitam $\frac{a}{b}$ quam quævis alia fractio minor ipſa $\frac{a}{b}$, quæ fractio minorem habeat denominatorem, ſeu quæ in terminis ſimplicioribus (quod idem eſt) concipiatur. Quæ de

de fractionibus intermediis inter $\frac{p}{p'}$, & $\frac{r}{r'}$ demonstravimus, locum etiam obtinent quoad fractiones intermedias inter $\frac{r}{r'}$, & $\frac{t}{t'}$, &c. si ϵ, &c. sint numeri unitate majores; quod ad aliam quoque seriem $\frac{q}{q'}$, $\frac{s}{s'}$, $\frac{u}{u'}$, &c. extenditur, si δ, ζ, &c. sint numeri unitate majores; tunc autem fractiones intermediæ singulæ majores quam $\frac{a}{b}$ successive imminuentur, & quantitatem ipsam $\frac{a}{b}$ accuratius exprīment præ alia fractione; quæ sit major eadem $\frac{a}{b}$, quæque in terminis simplicioribus concipiatur.

Quod si β sit numerus unitate major, fractionem $\frac{q}{q'}$ præcedere poterunt fractiones $\frac{p+1}{1}$, $\frac{2p+1}{2}$, $\frac{3p+1}{3}$, &c. usque ad $\frac{\beta p+1}{\beta} = \frac{q}{q'}$, & hæ fractiones iisdem proprietatibus perinde ac reliquæ intermediæ gaudebunt. *Quod erat* &c.

 784.

784. *Coroll.* 1. Duæ igitur haberi poſſunt completæ fractionum ſeries quantitatem $\frac{a}{b}$ verſus convergentes, nimirum

(1.ª) *continens fractiones creſcentes minores quam* $\frac{a}{b}$

$$\frac{p}{p'};\ \frac{q+p}{q'+p'};\ \frac{2q+p}{2q'+p'};\ \frac{3q+p}{3q'+p'}\ \ldots\ \frac{\gamma q+p}{\gamma q'+p'};\ \frac{r}{r'};$$

$$\frac{s+r}{s'+r'};\ \frac{2s+r}{2s'+r'};\ \frac{3s+r}{3s'+r'}\ \ldots\ldots\ \frac{\varepsilon s+r}{\varepsilon s'+r'};\ \frac{t}{t'};$$

$$\frac{u+t}{u'+t'};\ \text{\&c. \&c. \&c.}$$

(2.ª) *continens fractiones decreſcentes majores quam* $\frac{a}{b}$

$$\frac{p+1}{1};\ \frac{2p+1}{2};\ \frac{3p+1}{3}\ \ldots\ \frac{\beta p+1}{\beta};\ \frac{q}{q'};\ \frac{r+q}{r'+q'};$$

$$\frac{2r+q}{2r'+q'};\ \frac{3r+q}{3r'+q}\ \ldots\ \frac{\delta r+q}{\delta r'+q'};\ \frac{s}{s'};\ \frac{t+s}{t'+s'};\ \text{\&c.\&c. \&c.}$$

785. *Coroll.* 2. Si quantitas $\frac{a}{b}$ fuerit irrationalis duæ ſeries præcedentes in infinitum excurrent, quod fractionum *primariarum* $\frac{p}{p'}$, $\frac{q}{q'}$, $\frac{r}{r'}$, &c. ſeries abit

in

in infinitum (763.), sed si eadem quantitas fuerit rationalis, dicta series erit finita [*l. cit.*], & ultima ejusdem fractio ipsi $\frac{a}{b}$ erit æqualis (767.). Ergo in ista fractione una ex duabus supra allatis seriebus $\frac{p}{p'}, \frac{r}{r'}, \frac{t}{t'}$, &c. $\frac{q}{q'}, \frac{s}{s'}, \frac{u}{u'}$, &c. finem habebit, sed altera infinitum versus excurrere poterit. Etenim finge, quantitatem δ esse ultimum fractionis continuæ denominatorem; erit $\frac{s}{s'}$ ultima ex fractionibus primariis, & series fractionum majorum quam $\frac{a}{b}$ in ipsa fractione $\frac{s}{s'}$ terminabit. Altera vero series fractionum minorum quam $\frac{a}{b}$ remanebit suspensa in fractione $\frac{r}{r'}$ fractionem $\frac{s}{s'}$ præcedente; ut itaque continuetur, animadvertendum, denominatorem ε, qui ultimum denominatorem δ subsequitur, fieri $= \infty$; quum enim fractio continua terminet in δ, ultimus ejusdem fractionis terminus erit $\delta + 0 = \delta + \frac{1}{\infty}$, adeoque si quantitati δ addere velimus $\frac{1}{\varepsilon}$, ut fiat $\delta + \frac{1}{\varepsilon}$, erit $\varepsilon = \infty$, quare fractio $\frac{t}{t'}$,

quæ in fractionum primariarum serie fractionem $\frac{s}{s'}$ subsequeretur, foret $\frac{\infty s + r}{\infty s' + r'} = \frac{s}{s'}$. Ex fractionum igitur intermediarum proprietate, ob $\varepsilon = \infty$, infinitæ fractiones intermediæ inter fractiones $\frac{r}{r'}$, & $\frac{t}{t'}$ interponi poterunt, nimirum $\frac{s + r}{s' + r'}$; $\frac{2s + r}{2s' + r'}$; $\frac{3s + r}{3s' + r'}$; &c. quæ post fractionem $\frac{r}{r'}$ incipientes in infinitum continuari poterunt.

786. *Coroll.* 3. Generatim igitur si sequens proponatur Problema: *Data fractione multo cyphrarum numero prædita; invenire in terminis minimis omnes fractiones, qui ad verum ejusdem fractionis valorem quantum fieri potest proxime accedant;*

(1.°) Reducatur fractio proposita in fractionem continuam (759.).

(2.°) Per numeros α, β, γ &c. inventos fiant fractiones $\frac{p}{p'}, \frac{q}{q'}, \frac{r}{r'}$, &c. (769.), quarum ultima eadem erit cum fractione proposita (767.).

(3.°) Fractiones istæ erunt per vices majores & minores fractione proposita $\frac{a}{b}$ (778.), eruntque in terminis successi-

cessive majoribus conceptæ. Sic omnes fractiones in terminis proposita simplicioribus conceptæ obtinebuntur, quæ singulæ problemati satisfacient.

(4.°) Quod si fractiones tam minores quam majores proposita singillatim considerare libeat, inter fractiones primarias omnes intermediæ possibiles interponantur, ac duæ formentur fractionum convergentium series, quarum prima omnes minores, secunda omnes majores quam proposita continebunt (780. 783.); quævis enim ex hisce seriebus similes cum fractionibus primariis $\frac{p}{p'}$, $\frac{q}{q'}$, $\frac{r}{r'}$, &c. habebit proprietates.

(5.°) Potest tamen accidere, ut una ex fractionibus intermediis unius seriei tam prope non accedat ad fractionis proposita valorem perinde ac fractio aliqua alterius seriei, quamvis in terminis minus quam ista simplicibus concipiatur; quamobrem fractiones intermedias non expedit adhibere, nisi oporteat, ut fractiones quæsitæ sint omnes minores, vel majores quam fractio proposita.

Exemplum. Ratio semiperipheriæ ad radium, sive p ad r, est ut 3 ad 1 proxime, sed accuratius (ut post immensos labores invenit *Ludolphus a Ceulen*) habetur $p:r = 3, 14159, 26535, 89793, 23846, 26433, 83279, 50288, 41971, 69399, \ldots$ &c. ad 1. Itaque

Numerus decimalis respiciens p commutetur secundum regulas traditas (759.) in fractionem continuam, ubi animadvertendum, pro majore termi-

 no-

norum rationem datam designantium numero exactiorem calculum proditurum. Quum autem figurarum omnium decimalem p constituentium in fractionem continuam reductio sit maxime defatigans, sat erit fractionem continuam, quæ primis 32 figuris respondeat, determinare. Invenientur igitur quoti 3, 7, 15, 1, 292, 1, 1, &c. quare proveniet

$$3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{292+\cfrac{1}{1+\cfrac{1}{1+\&c.}}}}}}$$

unde per fractiones æquivalentes (766.) elicitur series

$$\frac{3}{1},\ \frac{22}{7},\ \frac{333}{106},\ \frac{355}{113},\ \frac{103993}{33102},\ \frac{104348}{33215},\ \frac{208341}{66317},\ \&c.$$

Per primam fractionem fit $p:r = 3:1$, quæ ratio per numeros integros non majores accuratius exprimi nequit.

Per secundam fractionem habemus $p:r = 22:7$, quæ est ratio *Archimedea*.

Per quartam stat $p:r = 355:113$, quæ est ratio *Metiana* major vera, sed error minor est quam

$$\frac{1}{113 \times 33102},\ \text{verus enim valor inter}\ \frac{1}{113 \times 33102},$$

$$\&\ \frac{1}{113(113+33102)}\ \text{consistit (777.).}$$

O.

Omnes autem rationes per feriei terminos expresſæ, ſunt per vices minores ac majores quam vera; prima nempe eſt minor, ſecunda major, tertia minor, quarta major, & ita porro, prout paulo ante monuimus (778.).

787. Propoſitio VII. *Fractionem continuam in ſeriem, cujus termini quoad ſigna per vices progrediantur, reſolvere.*

Reſol. Quoniam fractiones æquivalentes, ſi a ſecunda primam ſubtrahas, a tertia ſecundam, a quarta tertiam, & ſic deinceps, jugiter imminuuntur (773.), pronum eſt, hinc ſeriem orituram ſignis alternis notatam, quæ tandem terminabit, terminante fractione continua, cujus idcirco valorem repræſentabit. Quare ſi ſumptis fractionibus æquivalentibus jam expoſitis (766.), fiant enunciatæ differentiæ, exurget ſeries

$$\frac{\alpha}{1}+\frac{b}{\beta}-\frac{bc}{\beta(\beta\gamma+c)}+\frac{bcd}{\delta(\beta\gamma+c)^2+\beta d(\beta\gamma+c)}-\frac{bcde}{\beta^2\varepsilon(\gamma\delta+d)^2+\delta e(\beta\gamma+c)^2+2\beta\delta\varepsilon c(\gamma\delta+d)+\beta de(\beta\gamma+c)+\delta^2\varepsilon c^2}+\&c.$$

quæ fractioni continuæ $\alpha+\cfrac{b}{\beta+\cfrac{c}{\gamma+\cfrac{d}{\delta+\&c.}}}$ æquivalebit.

Sic ſi regula iſta fractioni continuæ 1

$1 + \cfrac{1}{2 + \cfrac{1}{38 + \cfrac{1}{7 + \cfrac{1}{4}}}}$ applicetur, hæc fractio resolvetur in seriem $\frac{1}{2} - \frac{1}{154} + \frac{1}{41657} - \frac{1}{1212381}$; in qua demptis terminis negativis ab affirmativis, residuum nuper expositam exprimet fractionem continuam, adeoque ejusdem summam $\frac{3347}{2241}$ [766.].

788. Propositio VIII. *Seriem, cujus termini quoad signa per vices progrediuntur, in fractionem continuam convertere.*

Resolutio. Serie ex differentiis in Prop. præced. expositis revocata, hujus unusquisque terminus conferatur cum termino respondente, sive homologo propositæ seriei, quæcumque sit ejus forma, nam esse potest vel $x = A - B + C - D + E - F +$ &c.

vel $x = \frac{a}{A} - \frac{a}{AB} + \frac{a}{ABC} - \frac{a}{ABCD} +$ &c.

vel $x = A - By + Cy^2 - Dy^3 +$ &c. vel $x =$

$\frac{A}{P} - \frac{By}{Qz} + \frac{Cy^2}{Rz^2} - \frac{Dy^3}{Sz^3} +$ &c. &c.

Deinde acceptis pro libito fractionis continuæ genera-

ralis $\alpha + \cfrac{b}{\beta + \cfrac{c}{\gamma + \&c.}}$ numeratoribus b, c, d, &c.

per dictas comparationes definiantur denominatorum β, γ, δ, &c. valores, vel contra, dummodo numeratores arbitrarii sint tales, ut per ipsos denominatores evadant numeri integri, vel contra. Quibus peractis, substituantur in fractione continua generali numeratores pro b, c, d, &c. & denominatores pro β, γ, δ, &c. ac prodibit fractio continua quæsita; *Quod erat* &c.

789. *Coroll.* Quæcumque igitur functio in seriem per vices quoad signa progredientem resolubilis, in fractionem continuam converti potest.

Exemplum I. Esto ad fractionem continuam reducenda series $x = A - B + C - D + E - F + \&c.$ Hujus termini cum terminis seriei $x = \frac{b}{\beta} - \frac{bc}{\beta(\beta\gamma + c)}$ + &c. comparatis, habemus

(1.) $A = \frac{b}{\beta}$; $B = \frac{bc}{\beta(\beta\gamma + c)}$; $C =$

$\frac{bcd}{\delta(\beta\gamma + c)^2 + \beta d(\beta\gamma + c)}$; $D =$ &c. pariterque

(2.) $A = \frac{b}{\beta}$; $\frac{B}{A} = \frac{c}{\beta\gamma + c}$; $\frac{C}{B} = \frac{\beta d}{\delta(\beta\gamma + c) + \beta d}$; $\frac{C}{D}$

$=$ &c. Ex prima æquatione num. 2. fit $b = A\beta$;

Ex

Ex secunda eruitur $c = \frac{B\beta\gamma}{A-B}$; Ex tertia prodit $d = \frac{C\beta\gamma\delta + C\delta c}{\beta(B-C)}$, & restituendo valorem inventum ipsius c, provenit $d = \frac{C\beta\gamma\delta + \frac{BC\beta\gamma\delta}{A-B}}{\beta(B-C)} = \frac{AC\gamma\delta}{(A-B)(B-C)}$. Haud absimili modo reliqui numeratores e, f, &c. definientur. Sed denominatores β, γ, δ, &c. ita sunt eligendi, ut numeratores b, c, d, &c. fiant numeri integri; ergo pone $\beta = 1$, $\gamma = A - B$, $\delta = B - C$, &c. ac prodibit $b = A$, $c = B$, $d = AC$, &c. Suffice hos valores in fractione continua generali $x = \cfrac{b}{\beta + \cfrac{c}{\gamma + \cfrac{d}{\delta + \&c.}}}$

& obtinebis $x = A - B + C - D + \&c. =$

$$\cfrac{A}{1 + \cfrac{B}{A - B + \cfrac{AC}{B - C + \cfrac{BD}{C - D + \&c.}}}}$$

Exemplum II. Esto series $\frac{a}{A} - \frac{a}{AB} + \frac{a}{ABC} -$

$\frac{a}{ABCD}$ + &c. in fractionem continuam transformanda.

Facta supra enunciata conlatione, invenietur $\frac{a}{A} = \frac{b}{\beta}$; $\left(\frac{a}{AB} : \frac{a}{A}\right) = \frac{1}{B} = \frac{c}{\beta\gamma + c}$; $\left(\frac{a}{ABC} : \frac{a}{AB}\right) = \frac{1}{C} = \frac{\beta d}{\delta(\beta\gamma + c) + \beta d}$; &c.

Ex quibus elicies $b = \frac{a\beta}{A}$; $c = \frac{\beta\gamma}{B - 1}$; $d = \frac{B\delta\gamma}{(B-1)(C-1)}$; &c. Sed quum numeratores tales esse debeant, ut denominatores fiant quantitates integræ, vel vicissim, statue $\beta = A$, $\gamma = B - 1$, $\delta = C - 1$, &c. adeoque $b = a$, $c = B = A$, $d = B$, &c. quibus valoribus in fractionis continuæ formula generali subrogatis, exurget $x = \frac{a}{A} - \frac{a}{AB} + \frac{a}{ABC} - \frac{a}{ABCD} + \&c. =$

$$\cfrac{a}{A + \cfrac{A}{B - 1 + \cfrac{B}{C - 1 + \cfrac{C}{D - 1 + \&c.}}}}$$

Quod

Quod si fuerit $a = 2$, $A = B = C = D =$ &c. $= 3$, series proposita evadet $\frac{2}{3} - \frac{2}{9} + \frac{2}{27} - \frac{2}{81} +$ &c. quæ migrabit in fractionem continuam

$$\cfrac{2}{3 + \cfrac{3}{2 + \cfrac{3}{2 + \text{\&c.}}}}$$

CAPUT VI.

DE INVENIENDIS ÆQUATIONUM LIMITIBUS.

790. *Monitum.* Ut ad hoc argumentum accuratius redeam, quoniam mutato incognitæ signo, æquationis radices positivæ in negativas transeunt (443.), radices tantummodo positivas simplicitatis gratia primum considerabimus, deinde mutando signa in terminis, ubi incognita ad potestates impares attollitur, novæ hujus æquationis radices venient considerandæ, quæ cum signo negativo acceptæ, erunt radices negativæ æquationis propositæ.

791. Propositio I. *Sint* α, β, γ, &c. *radices reales positivæ, & inæquales æquationis propositæ. Sit differentiarum* $\alpha - \beta$, $\alpha - \gamma$, &c. *minima* $= \lambda$. *Si duo numeri* p, q *pro incognita in data æquatione substituantur, ita ut sit* $p > \alpha$, $q < \alpha$, *nec non* $p - q <$

$q < \lambda$, *notum est* (408.), *duas orituras quantitates*
$(p - \alpha)(p - \beta)(p - \gamma)$ &c.
$(q - \alpha)(q - \beta)(q - \gamma)$ &c.
Hisce stantibus, dico, duas istas quantitates signis contrariis affectas necessario prodituras.

Dem. Quum differentiæ $p-\alpha$, $q-\alpha$ signa habeant manifeste contraria, quantitates $p-\beta$, $p-\gamma$, &c. eodem signo cum suis respondentibus $q-\beta$, $q-\gamma$, &c. gaudebunt; si enim fuerit $\beta > p$, res patet, quia $q < p$; sin autem ponatur $\beta < p$; ob $p-q < \alpha-\beta$ [*per hyp.*], erit $\beta < \alpha - p + q$, ac multo magis $\beta < p - p + q$, seu $\beta < q$; Idem quoad ceteras radices ratiocinandum; quare quum $p-\alpha$, $q-\alpha$ habeant signa contraria, ceteræque quantitates $p-\beta$, $p-\gamma$, &c. $q-\beta$, $q-\gamma$, &c. eodem signo notentur, duæ dictæ quantitates ex substitutionibus ortæ duobus signis contrariis necessario afficientur. *Quod erat* &c.

792. *Coroll.* Sumpta igitur generatim $\Delta = \lambda$, vel $\Delta < \lambda$, si in data æquatione successive pro incognita x substituantur numeri progressionem arithmeticam $0, \Delta, 2\Delta, 3\Delta, 4\Delta$, &c. servantes; quæ ex hisce substitutionibus gradatim emergent, seriem efficient, in qua tot signorum aderunt variationes, quot radices habet æquatio reales, positivas & inæquales; ubi animadvertendum

(1.°) Quod si unicam æquatio habeat radicem realem positivam, vel si plures, quarum binarum differentias constet, non esse unitate minores, tunc quum sit $\Delta = \lambda = 1$, vel $\Delta = \lambda > 1$, accipi possunt numeri naturales $0, 1, 2, 3, 4$, &c. in æquatione pro x successive substituendi.

(2.°)

(2.°) Sed si hujusmodi radicum differentiæ sint unitate minores, vel si ignoretur, utrum sint tales, an contra, tunc sumi debet $\Delta = \lambda < 1$, sed ita, ut sit æqualis minimæ differentiæ λ, vel eadem minor.

Tota ergo difficultas eo deducta est, ut inveniatur valor, quem exigit Δ, ne superet minimam differentiam λ, quod problematis sequentis objectum constituit.

793. Propositio II. *Datam æquationem in aliam transformare, cujus radices sint differentiæ radicum æquationis propositæ.*

Resolutio. Hoc Problema, quod ad Cap. II. Sect. III. propriæ pertinet, huc transferre, Tironum commodo consulens, opportunum duxi. Sit itaque proposita æquatio

$$x^m - Ax^{m-1} + Bx^{m-2} - Cx^{m-3} + \&c. = 0 \ldots (A)$$

quoniam x quamlibet radicem indiscriminatim repræsentat, esto x' alia quævis ex istis radicibus. Fiet itidem

$$x'^m - Ax'^{m-1} + Bx'^{m-2} - Cx'^{m-3} + \&c. = 0 \ldots (B).$$

Ponatur radicum differentia $x' - x = u$; erit $x' = x + u$, quo valore pro x' in secundam æquationem (B) introducto, & transformata, quæ hinc exurgit, ordinata per u (colligendo nempe terminos, ubi primo non adest u, deinde ubi adest, tum ubi adest u^2, & sic deinceps), proveniet (446.)

x^m

$x^m - Ax^{m-1} + Bx^{m-2} - Cx^{m-3} +$ &c. $= X$.

$+ mx^{m-1}u - (m-1)Ax^{m-2}u + (m-2)Bx^{m-3}u$
$-$ &c. $= Yu$.

$+ \frac{m(m-1)}{2} x^{m-2}u^2 - \frac{(m-1)(m-2)}{2} Ax^{m-3}u^2 +$
&c. $= Zu^2$.
&c. usque ad ultimum terminum u^m.

Seu brevius, $X + Yu + Zu^2 + Vu^3 +$ &c. $+ u^m$. Sed $X = 0$; ergo dividendo per u, transformata evadet

$Y + Zu + Vu^2 +$ &c. $+ u^{m-1} = 0 \ldots\ldots$ (C)
Quod erat &c.

794. *Coroll.* 1. Si propositæ (*A*) radices sint α, β, γ, &c. quarum numerus $= m$, fiet $u = \alpha - \beta, \alpha - \gamma$, &c. $\beta - \alpha, \beta - \gamma$, &c. $\gamma - \alpha, \gamma - \beta$, &c.; quælibet igitur istiusmodi classis exhibebit differentiarum numerum $= m - 1$, & omnium differentiarum summa erit $m(m-1)$, adeoque transformata (C) erit semper paris dimensionis, quia $m(m-1)$, ad quem gradum ejus dimensio ascendere debet, est semper numerus par; quapropter

795. *Coroll.* 2. Quum transformatæ radices sint $\alpha - \beta, \beta - \alpha, \alpha - \gamma, \gamma - \alpha$, &c. ipsa transformata plura continebit radicum paria signis tantum inter se diversa, ex. gr. $f, -f, g, -g$, &c. (565.); quare terminis in locis paribus, seu omnibus incognitæ potestatibus imparibus instar derivativarū (556.) carebit.

796. *Coroll.* 3. Ergo si fiat $\frac{m(m-1)}{2} = n$, vel $m(m-1) = 2n$, æquatio transformata evadet

$u^{2n} - au^{2(n-1)} + bu^{2(n-2)} - cu^{2(n-3)} + \&c. = 0$, & statuendo $u^2 = v$, nullus terminus in dicta transformata depressa

$v^n - av^{n-1} + bv^{n-2} - cv^{n-3} + \&c. = 0 \ldots\ldots (D)$ deficiet, ubi radices (quadrata nempe $(\alpha - \beta)^2$, $(\alpha - \gamma)^2$, &c.) sunt omnes reales positivæ, & dimensio est dimidium dimensionis, quam habet transformata (C), adeoque ista æquatio (D) habet radices $(\alpha - \beta)^2$, $(\alpha - \gamma)^2$, &c. non vero $(\beta - \alpha)^2$, $(\gamma - \alpha)^2$, &c. vel viciſſim.

797. *Coroll.* 4. Coëfficiens igitur a continebit omnium quadratorum $(\alpha - \beta)^2$, $(\alpha - \gamma)^2$, &c. summam (413.), singula enim ista quadrata præbent valorem incognitæ v. Est autem $(\alpha - \beta)^2 + (\alpha - \gamma)^2 + (\beta - \gamma)^2 + \&c = (m-1)(\alpha^2 + \beta^2 + \gamma^2 + \&c.) - 2(\alpha\beta + \alpha\gamma + \beta\gamma + \&c.)$, quia si radices α, β, γ sint tantummodo tres, ex quadratis $(\alpha - \beta)^2, (\alpha - \gamma)^2$, $(\beta - \gamma)^2$ emergere nequeunt, nisi quadrata $(3-1)(\alpha^2 + \beta^2 + \gamma^2)$, adeoque si radicum numerus sit m, hujusmodi quadratorum numerus erit $m-1$, quod etiam valet de altioribus potestatibus. Ergo quoniam $\alpha\beta + \alpha\gamma + \beta\gamma + \&c. = B$, & $\alpha^2 + \beta^2 + \gamma^2 + \&c. = A^2 - 2B$ (433.), erit $a = (m-1)A^2 - 2mB + 2B - 2B = (m-1)A^2 - 2mB$. Eodem modo invenientur aliorum coëfficientium b, c, &c. valores.

798. *Scholion.* Ut istos valores facilius adsequamur ponamus

$A' = \alpha + \beta + \gamma +$ &c.
$A'' = \alpha^2 + \beta^2 + \gamma^2 +$ &c.
$A''' = \alpha^3 + \beta^3 + \gamma^3 +$ &c.
$A^{IV} = \alpha^4 + \beta^4 + \gamma^4 +$ &c.
&c. &c.

Ex dictis (*l. cit.*) erit

$A' = A$
$A'' = AA' - 2B$
$A''' = AA'' - BA' + 3C$
$A^{IV} = AA''' - BA'' + CA' - 4D$
$A^{V} = AA^{IV} - BA''' + CA'' - DA' + 5E$
&c. &c.

qui valores facile reperiuntur, si pro numeri cit. coëfficientibus $p, q, r, s, t,$ &c. sumantur singillatim præsentes $A, B, C, D, E,$ &c. ibi enim ex. gr. est $C = Bp - Aq + 3r$; sed q (summa quadratorum hic indicata per B) est A''; ergo quod illic C, hic est $AA'' - BA' + 3C$; & sic de reliquis, quorum valorum lex progressiva facile detegitur.

Ponamus præterea

$a' = (\alpha - \beta)^2 + (\alpha - \gamma)^2 + (\beta - \gamma)^2 +$ &c.
$a'' = (\alpha - \beta)^4 + (\alpha - \gamma)^4 + (\beta - \gamma)^4 +$ &c.
$a''' = (\alpha - \beta)^6 + (\alpha - \gamma)^6 + (\beta - \gamma)^6 +$ &c.
$a^{IV} = (\alpha - \beta)^8 + (\alpha - \gamma)^8 + (\beta - \gamma)^8 +$ &c.
&c. &c.

Vidimus autem esse $a' = (\alpha - \beta)^2 + (\alpha - \gamma)^2 + (\beta - \gamma)^2 +$ &c. $= (m - 1)(\alpha^2 + \beta^2 + \gamma^2 +$ &c.$) - 2(\alpha\beta + \alpha\gamma + \beta\gamma +$ &c.$)$; sed $\alpha^2 + \beta^2 + \gamma^2 +$ &c. $= A''$, atque $\alpha\beta + \alpha\gamma + \beta\gamma +$ &c. $= B$; ergo $a' = (m - 1) A''$

 —

$- 2B$. At ex formula $A'' = AA' - 2B$ elicitur $2B - AA' - A'' = (A')^2 - A''$; ergo $a' = (m-1)A'' - \frac{2((A')^2 - A'')}{2}$.

Si $\alpha - \beta, \alpha - \gamma, \beta - \gamma$, &c. eleventur ad quartam potestatem obtinebitur $a'' = (\alpha - \beta)^4 + (\alpha - \gamma)^4 + (\beta - \gamma)^4 +$ &c. $= (m-1)(\alpha^4 + \beta^4 + \gamma^4) - 4(\alpha^3\beta + \alpha^3\gamma + \&c.) + 6(\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 + \&c.) - 4(\alpha\beta^3 + \alpha\gamma^3 + \beta\gamma^3 + \&c.) = (m-1)(\alpha^4 + \beta^4 + \gamma^4 + \&c.) - 4(\alpha\beta^3 + \alpha\gamma^3 + \beta\gamma^3 + \alpha^3\beta + \alpha^3\gamma + \beta^3\gamma + \&c.) + 6(\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 + \&c.)$ Sed $\alpha^4 + \beta^4 + \gamma^4 + \&c. = A^{IV}$; nec non $\alpha\beta^3 + \alpha\gamma^3 + \beta\gamma^3 + \&c. = (\alpha + \beta + \gamma + \&c.)(\alpha^3 + \beta^3 + \gamma^3 + \&c.) - \alpha^4 - \beta^4 - \gamma^4 - \&c. = A'A''' - A^{IV}$, pariterque $\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 + \&c. = \frac{(\alpha^2 + \beta^2 + \gamma^2 + \&c.)^2}{2} - \frac{\alpha^4 - \beta^4 - \gamma^4 - \&c.}{2} = \frac{(A'')^2 - A^{IV}}{2}$; Ergo $a'' = (m-1)A^{IV} - 4(A'A''' - A^{IV}) + 6\frac{((A'')^2 - A^{IV})}{2}$. Quare sic prosequendo habetur

$$a' = (m-1)A'' - 2\frac{((A')^2 - A'')}{2}$$

$$a'' = (m-1)A^{IV} - 4(A'A''' - A^{IV}) + 6\frac{((A'')^2 - A^{IV}}{2}$$

a'''

$a^{\prime\prime\prime} = (m-1)A^{VI} - 6(A^{\prime}A^{V} - A^{VI}) + 15(A^{\prime\prime}A^{IV} - A^{VI})$

$- 20\,\frac{((A^{\prime\prime\prime})^2 - A^{VI})}{2}$

&c. &c.

sive

$a^{\prime} = mA^{\prime\prime} - \frac{2(A^{\prime})^2}{2}$

$a^{\prime\prime} = mA^{IV} - 4A^{\prime}A^{\prime\prime\prime} + \frac{6(A^{\prime\prime})^2}{2}$

$a^{\prime\prime\prime} = mA^{VI} - 6A^{\prime}A^{V} + 15A^{\prime\prime}A^{IV} - \frac{20((A^{\prime\prime\prime})^2}{2}$

&c. &c.

Generatim ponendo μ pro indeterminato accentuum numero, erit $a^{\mu} = mA^{\prime\prime\mu} - 2\mu A^{\prime}(A^{\prime\prime\mu-1}) + \frac{2\mu(2\mu-1)}{1\,.\,2}A^{\prime\prime}(A^{\prime\prime\mu-2}) - \&c. \pm \frac{2\mu(2\mu-1)(2\mu-2)\ldots\ldots(\mu+1)}{1\,.\;2\,.\;3\;\ldots\;\mu}\,.\,\frac{(A^{\mu})^2}{2}$, ubi ultimus factor accipiendus esse debet $= \frac{\mu+1}{\mu}$.

Sic pro a^{IV} (cum nempe $\mu = 4$) stabit

$a^{\mu} = mA^{\prime\prime\mu} - 2\mu A^{\prime}(A^{\prime\prime\mu-1}) + \frac{2\mu(2\mu-1)}{1\,.\;2}A^{\prime\prime}(A^{\prime\prime\mu-2} - \frac{2\mu(2\mu-1)(2\mu-2)}{1\,.\;2\,.\;3}A^{\prime\prime\prime}(A^{\prime\prime\mu-3}) +$

$\frac{2u(2u-1)(2u-2)(2u-3)}{1.\ 2.\ 3.\ 4}\ \frac{(A^{\mu})^2}{2}$; seu $a^{IV} = mA^{VIII} - 8A'A^{VII} + 28A''A^{VI} - 56A'''A^{V} + 740\ \frac{(A^{IV})^2}{2}$; & sic de ceteris.

Cognitis quantitatum a', a'', a''', &c. valoribus, coëfficientium a, b, c, &c. æquationis (D) valores innotescent per formulas subsequentes ex formulis nuper traditis erutas, exhibendo per A, A'', A''', &c. quantitatum B, C, D, &c. valores, litteris majusculis in minusculas commutatis, nimirum

$a = a'$

$b = \frac{aa' - a''}{2}$

$c = \frac{ba' - aa'' + a'''}{3}$

$d = \frac{ca' - ba'' + aa'''}{4}$

&c. &c.

nam $A' = A = a$.

$A'' = AA' - 2b$; nempe $a'' = aa' - 2b$; hinc $b = \frac{aa' - a''}{2}$.

$A''' = AA'' - BA' + 3C$; nimirum $a''' = aa'' - ba' + 3c$; hinc $c = \frac{ba' - aa'' + a'''}{3}$.

A^{IV}

$A^{\text{iv}} = AA''' - BA'' + CA' - 4D$; hoc est $a^{\text{iv}} = aa''' - ba'' + ca' - 4d$; unde $d = \frac{ca' - ba'' + aa''' - a^{\text{iv}}}{4}$. &c.

Inventis igitur per æquationis datæ $[A]$ vel $[B]$ coëfficientes cognitos A, B, C; &c. valoribus quantitatum A', A'', A''', &c. usque ad A, in qua accentuum numerus sit $= 2n$, per istos quærantur valores quantitatum a'; a'', a''', &c. usque ad a, in qua accentuum numerus sit $= n$; in penultima enim formula quantitati a' respondet A'', quantitati a'' respondet A^{iv}, &c. ut illam intuenti palam erit; ac tandem per postremos hosce valores invenientur valores quæsiti coëfficientium a, b, c, &c.

799. *Coroll.* 5. Si in æquatione $[D]$ mutetur u in $-u$, ipsa eadem persistet; quum enim contineat quadrata differentiarum ex radicibus propositæ $[B]$; perinde erit tam si ponamus, eam constare quadratis $(\alpha - \beta)^2$, $(\alpha - \gamma)^2$, $(\beta - \gamma)^2$, &c. quam si statuamus, ipsam efformari quadratis $(\beta - \alpha)^2$, $(\gamma - \alpha)^2$, $(\gamma - \beta)^2$, &c. mutando igitur signa in locis paribus in æquatione $[B]$, æquatio $[D]$ mutationem non patitur.

800. *Coroll.* 6. Dicta æquatio $[D]$ eadem pariter persistet, si omnes æquationis $[B]$ radices augeantur, vel imminuantur eadem quantitate, quæcumque ea sit; nam $(\alpha \pm \pi) - (\beta \pm \pi) = \alpha - \beta$, &c. adeoque $\overline{(\alpha \pm \pi) - (\beta \pm \pi)}^{\,2} = (\alpha - \beta)^2$, &c. Hinc si æquatio $[B]$ secundo termino gaudeat, potest hic eliminari, ac deinde construi, ut diximus, æquatio $[D]$; quæ eadem permanebit ac si dictus secundus termi-

nus non evanuisset. At hujus termini exturbatio id habet commodi, quod coëfficientium a, b, c, &c. perquisitionem reddit faciliorem, fit enim $A = 0$, ac proinde $A' = 0$; formulæ igitur nuperæ evadent

$A' = 0$	$a' = mA''$	$a = a'$
$A'' = -2B$	$a'' = mA^{IV} + 6\frac{(A'')^2}{2}$	$b = \frac{aa' - a''}{2}$
$A''' = 3C$		
$A^{IV} = -BA'' - 4$	$a''' = mA^{VI} + 15A''A^{IV} - 20\frac{(A''')^2}{2}$	$c = \frac{ba' - a'' + a'''}{3}$
&c.	&c.	&c.

801. *Coroll.* 7. Quoniam æquationis [D] radices sunt quadrata differentiarum $\alpha - \beta, \alpha - \gamma, \beta - \gamma$, &c. si æquationis hujusce signa sint omnia similia, ejus radices forent quadrata negativa; sed quadratum negativum est impossibile; ergo differentiæ inter radices æquationis [B] forent omnes imaginariæ, velut $t\sqrt{-1}$, vel $-t\sqrt{-1}$.

802. *Coroll.* 8. Si æquatio [B] unum vel plures radicum æqualium paria possideat, æquatio [D] unum, vel plures habebit quantitatis v valores $= 0$. Hinc methodus explorandi, an æquatio data habeat nec ne radices invicem æquales; quod si tales habeat, illæ ex dictis (514.) eliciantur, & æquatio per debitam divisionem deprimatur.

803. Propositio III. *Radicum æquationis datæ* (A) *vel* (B) *Prop. antec. limites invenire.*

Resolutio. Posita $r = n$ si æquatio data nullas, vel $r < n$, si per divisionem depressa nonnullas prius habuerit radices invicem æquales, æquationis termini invertantur, ita ut ipsa evadat

$$1 + \alpha v + \beta v^2 + \gamma v^3 + \text{\&c.} \ldots\ldots + \pi v^r = 0 \ldots (F).$$

Tum

Tum ſtatue $v = \frac{1}{y}$, & æquatione (E) ordinata per y, habebitur

$$y^r + \alpha y^{r-1} + \beta y^{r-2} + \gamma y^{r-3} + \&c. \ldots + \pi = 0 \ldots (F).$$

Quæratur limes l radicum hac in æquatione degentium, ita ut l ſuperet valores quoſcumque poſitivos incognitæ y (467.). Ergo quoniam $l > y$, erit $\frac{1}{l} < \frac{1}{y}$, ſeu $\frac{1}{l} < v$, adeoque ob $v = u^2$ (796.), erit $\frac{1}{l}$ minor quovis valore quantitatis u^2, proptereaque $\frac{1}{\sqrt{l}} < u$, nimirum $\frac{1}{\sqrt{l}}$ neceſſario minor qualibet ex differentiis, quæ intercedunt inter radices reales póſitivas, & inæquales propoſitæ (B). Hinc

(1.°) Si denominator $\sqrt{l} = 1$, vel ſi $\sqrt{l} < 1$, certum erit, nullas in æquatione (B) adeſſe radices reales, quarum differentiæ ſint unitate minores, quia $\frac{1}{\sqrt{l}}$ vel ſit $= 1$, vel > 1. Hoc igitur in caſu ſtatui procul dubio poteſt $\Delta = 1$ (792. n. (1.)).

(2.°) Sin autem denominator $\sqrt{l} > 1$, tunc ob fractionem $\frac{1}{\sqrt{l}}$ fieri poteſt, ut æquatio (B) radices

ha-

habeat, quarum differentiæ sint unitate minores. Sed quum harum differentiarum minima λ, sive u, sit semper $> \frac{1}{\sqrt{l}}$; accipi semper poterit $\Delta = \frac{1}{\sqrt{l}}$, vel $\Delta < \frac{1}{\sqrt{l}}$ [*l. cit.*], esse enim debet $\Delta < u$; vel $\Delta = u$.

Generatim si $\sqrt{l}$ extrahi nequeat, quum radicales in substitutionibus locum non habeant, sumatur numerus proxime major quam $\sqrt{l}$, qui fiat $= K$, idemque peragatur, si $\sqrt{l}$ sit numerus integer. Erit in utroque casu $\frac{1}{K} = \frac{1}{\sqrt{l}}$; adeoque sumi semper poterit $\Delta = \frac{1}{K}$, vel $\Delta < \frac{1}{K}$. Invento igitur æquationis (F) limite l, sumptoque K proxime majore quam $\sqrt{l}$, fiat $\Delta = \frac{1}{K}$; tum numeris 0, $\frac{1}{K}$, $\frac{2}{K}$, $\frac{3}{K}$, &c. pro incognita in æquatione successive suffectis, quæ ex hac operatione prodibunt, seriem formabunt, in qua tot aderunt signorum variationes, quot radices reales positivas & inæquales æquatio comprehendet, & istæ radices inter duos dictæ seriei numeros continuos signis diversis affectos (ex. gr. inter $\frac{b}{K}$, & $\frac{b+1}{K}$) delitescent; quare numerus in-

integer, qui alterutro ex valoribus $\frac{b}{K}$, $\frac{b+1}{K}$ appropinquabit, erit valor integer proximus unius ex radicibus quæsitis. *Quod erat* &c.

804. *Scholion.* Ad fractionum, substitutionumque molestiam imminuendam expedit (1.°) ut pro incognitæ x potestatibus, scribantur in æquatione data potestates fractionis $\frac{x}{K}$, ac deinde per ipsius K maximam potestatem omnes æquationis termini multiplicentur; sic fractiones vitabuntur; (2.°) ut substitutis successive numeris incognitæ $x = 0, 1, 2, 3$, &c. respondentibus, ex primis quantitatibus $m+1$, quæ sic agendo proveniunt, fiat transitus prius per differentias primas, secundas, tertias, &c. tum per additiones, ad reliquos numeros inveniendos. Exemplo res illustratur.

Esto $x^3 - 63x + 189 = 0$.

Numeri substitutionibus $x = 0, 1, 2, 3$ respondentes erunt 189, 127, 71, 27. Subtrahe 189 ex 127, & residuum -62 superponatur numero 189. Ex 71 deme 127, & residuum -56 collocetur supra 127, & ita porro. Primas hasce differentias excipiant secundæ, subtrahendo -62 ex -56, & residuum 6 scribatur supra -62; pari modo subducto -56 ex -44, residuum 12 notetur supra -56. Eodem ordine deveniatur ad tertiam differentiam, subtrahendo 6 ex 12, & ponendo 6 supra 6. Quibus

(2.°) Sed si hujusmodi radicum differentiæ sint unitate minores, vel si ignoretur, utrum sint tales, an contra, tunc sumi debet $\Delta = \lambda < 1$, sed ita, ut sit æqualis minimæ differentiæ λ, vel eadem minor.

Tota ergo difficultas eo deducta est, ut inveniatur valor, quem exigit Δ, ne superet minimam differentiam λ, quod problematis sequentis objectum constituit.

793. Propositio II. *Datam æquationem in aliam transformare, cujus radices sint differentiæ radicum æquationis propositæ.*

Resolutio. Hoc Problema, quod ad Cap. II. Sect. III. propriæ pertinet, huc transferre, Tironum commodo consulens, opportunum duxi. Sit itaque proposita æquatio

$$x^m - Ax^{m-1} + Bx^{m-2} - Cx^{m-3} + \&c. = 0 \ldots .(A)$$

quoniam x quamlibet radicem indiscriminatim repræsentat, esto x' alia quævis ex istis radicibus. Fiet itidem

$$x'^m - Ax'^{m-1} + Bx'^{m-2} - Cx'^{m-3} + \&c. = 0 \ldots .(B).$$

Ponatur radicum differentia $x' - x = u$; erit $x' = x + u$, quo valore pro x' in secundam æquationem (B) introducto, & transformata, quæ hinc exurgit, ordinata per u (colligendo nempe terminos, ubi primo non adest u, deinde ubi adest, tum ubi adest u^2, & sic deinceps), proveniet (446.)

x^m

$x^m - Ax^{m-1} + Bx^{m-2} - Cx^{m-3} +$ &c. $= X.$

$+ mx^{m-1}u - (m-1)Ax^{m-2}u + (m-2)Bx^{m-3}u$
$-$ &c. $= Yu.$

$+ \frac{m(m-1)}{2}x^{m-2}u^2 - \frac{(m-1)(m-2)}{2}Ax^{m-3}u^2 +$

&c. $= Zu^2.$
&c. usque ad ultimum terminum u^m.

Seu brevius, $X + Yu + Zu^2 + Vu^3 +$ &c. $+ u^m$. Sed $X = 0$; ergo dividendo per u, transformata evadet

$Y + Zu + Vu^2 +$ &c. $+ u^{m-1} = 0 \ldots\ldots$ (C)
Quod erat &c.

794. *Coroll.* 1. Si propositæ (A) radices sint α, β, γ, &c. quarum numerus $= m$, fiet $u = \alpha - \beta, \alpha - \gamma$, &c. $\beta - \alpha, \beta - \gamma$, &c. $\gamma - \alpha, \gamma - \beta$, &c.; quælibet igitur istiusmodi classis exhibebit differentiarum numerum $= m - 1$, & omnium differentiarum summa erit $m(m-1)$, adeoque transformata (C) erit semper paris dimensionis, quia $m(m-1)$, ad quem gradum ejus dimensio ascendere debet, est semper numerus par; quapropter

795. *Coroll.* 2. Quum transformatæ radices sint $\alpha - \beta, \beta - \alpha, \alpha - \gamma, \gamma - \alpha$, &c. ipsa transformata plura continebit radicum paria signis tantum inter se diversa, ex. gr. $f, -f, g, -g$, &c. (565.); quare terminis in locis paribus, seu omnibus incognitæ potestatibus imparibus instar derivativarũ (556.) carebit.

796. *Coroll.* 3. Ergo si fiat $\frac{m(m-1)}{2} = n$, vel $m(m-1) = 2n$, æquatio transformata evadet

$$u^{2n} - au^{2(n-1)} + bu^{2(n-2)} - cu^{2(n-3)} + \&c. = 0,$$

& statuendo $u^2 = v$, nullus terminus in dicta transformata depressa

$$v^n - av^{n-1} + bv^{n-2} - cv^{n-3} + \&c. = 0 \ldots\ldots (D)$$

deficiet, ubi radices (quadrata nempe $(\alpha - \beta)^2$, $(\alpha - \gamma)^2$, &c.) sunt omnes reales positivæ, & dimensio est dimidium dimensionis, quam habet transformata (C), adeoque ista æquatio (D) habet radices $(\alpha - \beta)^2$, $(\alpha - \gamma)^2$, &c. non vero $(\beta - \alpha)^2$, $(\gamma - \alpha)^2$, &c. vel vicissim.

797. *Coroll.* 4. Coëfficiens igitur a continebit omnium quadratorum $(\alpha - \beta)^2$, $(\alpha - \gamma)^2$, &c. summam (413.), singula enim ista quadrata præbent valorem incognitæ v. Est autem $(\alpha - \beta)^2 + (\alpha - \gamma)^2 + (\beta - \gamma)^2 + \&c. = (m-1)(\alpha^2 + \beta^2 + \gamma^2 + \&c.) - 2(\alpha\beta + \alpha\gamma + \beta\gamma + \&c.)$, quia si radices α, β, γ sint tantummodo tres, ex quadratis $(\alpha - \beta)^2$, $(\alpha - \gamma)^2$, $(\beta - \gamma)^2$ emergere nequeunt, nisi quadrata $(3-1)(\alpha^2 + \beta^2 + \gamma^2)$, adeoque si radicum numerus sit m, hujusmodi quadratorum numerus erit $m-1$, quod etiam valet de altioribus potestatibus. Ergo quoniam $\alpha\beta + \alpha\gamma + \beta\gamma + \&c. = B$, & $\alpha^2 + \beta^2 + \gamma^2 + \&c. = A^2 - 2B$ (433.), erit $a = (m-1)A^2 - 2mB + 2B - 2B = (m-1)A^2 - 2mB$. Eodem modo invenientur aliorum coëfficientium b, c, &c. valores.

798.

798. *Scholion.* Ut istos valores facilius adsequamur ponamus

$A' = \alpha + \beta + \gamma +$ &c.
$A'' = \alpha^2 + \beta^2 + \gamma^2 +$ &c.
$A''' = \alpha^3 + \beta^3 + \gamma^3 +$ &c.
$A^{IV} = \alpha^4 + \beta^4 + \gamma^4 +$ &c.
&c. &c.

Ex dictis (*l. cit.*) erit

$A' = A$
$A'' = AA' - 2B$
$A''' = AA'' - BA' + 3C$
$A^{IV} = AA''' - BA'' + CA' - 4D$
$A^{V} = AA^{IV} - BA''' + CA'' - DA' + 5E$
&c. &c.

qui valores facile reperiuntur, si pro numeri cit. coëfficientibus $p, q, r, s, t,$ &c. sumantur singillatim præsentes $A, B, C, D, E,$ &c. ibi enim ex. gr. est $C = Bp - Aq + 3r$; sed q (summa quadratorum hic indicata per B) est A''; ergo quod illic C, hic est $AA'' - BA' + 3C$; & sic de reliquis, quorum valorum lex progressiva facile detegitur.

Ponamus præterea

$a' = (\alpha - \beta)^2 + (\alpha - \gamma)^2 + (\beta - \gamma)^2 +$ &c.
$a'' = (\alpha - \beta)^4 + (\alpha - \gamma)^4 + (\beta - \gamma)^4 +$ &c.
$a''' = (\alpha - \beta)^6 + (\alpha - \gamma)^6 + (\beta - \gamma)^6 +$ &c.
$a^{IV} = (\alpha - \beta)^8 + (\alpha - \gamma)^8 + (\beta - \gamma)^8 +$ &c.
&c. &c.

Vidimus autem esse $a' = (\alpha - \beta)^2 + (\alpha - \gamma)^2 + (\beta - \gamma)^2 +$ &c. $= (m - 1)(\alpha^2 + \beta^2 + \gamma^2 +$ &c.$) - 2(\alpha\beta + \alpha\gamma + \beta\gamma +$ &c.$)$; sed $\alpha^2 + \beta^2 + \gamma^2 +$ &c. $= A''$, atque $\alpha\beta + \alpha\gamma + \beta\gamma +$ &c. $= B$; ergo $a' = (m - 1) A''$ —

$-2B$. At ex formula $A'' = AA' - 2B$ elicitur $2B = AA' - A'' = (A')^2 - A''$; ergo $a' = (m-1)A'' - \frac{2((A')^2 - A'')}{2}$.

Si $\alpha - \beta, \alpha - \gamma, \beta - \gamma$, &c. eleventur ad quartam potestatem obtinebitur $a'' = (\alpha-\beta)^4 + (\alpha-\gamma)^4 + (\beta-\gamma)^4 +$ &c. $= (m-1)(\alpha^4 + \beta^4 + \gamma^4) - 4(\alpha^3\beta + \alpha^3\gamma +$ &c.$) + 6(\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 +$ &c.$) - 4(\alpha\beta^3 + \alpha\gamma^3 + \beta\gamma^3 +$ &c.$) = (m-1)(\alpha^4 + \beta^4 + \gamma^4 +$ &c.$) - 4(\alpha\beta^3 + \alpha\gamma^3 + \beta\gamma^3 + \alpha^3\beta + \alpha^3\gamma + \beta^3\gamma +$ &c.$) + 6(\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 +$ &c.$)$ Sed $\alpha^4 + \beta^4 + \gamma^4 +$ &c. $= A^{IV}$; nec non $\alpha\beta^3 + \alpha\gamma^3 + \beta\gamma^3 +$ &c. $= (\alpha + \beta + \gamma +$ &c.$)(\alpha^3 + \beta^3 + \gamma^3 +$ &c.$) - \alpha^4 - \beta^4 - \gamma^4 -$ &c. $= A'A''' - A^{IV}$, pariterque $\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 +$ &c. $= \frac{(\alpha^2 + \beta^2 + \gamma^2 + \&c.)^2}{2} - \frac{\alpha^4 - \beta^4 - \gamma^4 - \&c.}{2} = \frac{(A'')^2 - A^{IV}}{2}$; Ergo $a'' = (m-1)A^{IV} - 4(A'A''' - A^{IV}) + 6\frac{((A'')^2 - A^{IV})}{2}$. Quare sic prosequendo habetur

$$a' = (m-1)A'' - 2\frac{((A')^2 - A'')}{2}$$

$$a'' = (m-1)A^{IV} - 4(A'A''' - A^{IV}) + 6\frac{((A'')^2 - A^{IV}}{2}$$

a'''

$a''' = (m-1)A^{VI} - 6(A'A^{V} - A^{VI}) + 15(A''A^{IV} - A^{VI})$

$- 20\ \frac{((A''')^2 - A^{VI})}{2}$

&c. &c.

five

$a' = mA'' - \frac{2(A')^2}{2}$

$a'' = mA^{IV} - 4A'A''' + \frac{6(A'')^2}{2}$

$a''' = mA^{VI} - 6A'A^{V} + 15A''A^{IV} - \frac{20((A''')^2}{2}$

&c. &c.

Generatim ponendo μ pro indeterminato accentuum numero, erit $a^{\mu} = mA^{''\mu} - 2\mu A'(A^{''\mu - 1}) + \frac{2\mu(2\mu-1)}{1.2} A''(A^{''\mu - 2}) -$ &c. $\pm \frac{2\mu(2\mu-1)(2\mu-2).....(\mu+1)}{1.\ 2.\ 3\ \ \mu} . \frac{(A^{\mu})^2}{2}$, ubi ultimus factor accipiendus esse debet $= \frac{\mu+1}{\mu}$.

Sic pro a^{IV} (cum nempe $\mu = 4$) stabit

$a^{\mu} = mA^{''\mu} - 2\mu A'(A^{\mu - 1}) + \frac{2\mu(2\mu-1)}{1.\ 2} A''$

$(A^{''\mu - 2} - \frac{2\mu(2\mu-1)(2\mu-2)}{1.\ 2.\ 3} A'''(A^{''\mu - 3}) +$

$\frac{2\mu(2\mu-1)(2\mu-2)(2\mu-3)}{1.\ 2.\ 3.\ 4}\ \frac{(A^{\mu})^2}{2}$; seu $a^{IV} =$ $m A^{VIII} - 8 A' A^{VII} + 28 A'' A^{VI} - 56 A''' A^{V} + 740 \frac{(A^{IV})^2}{2}$; & sic de ceteris.

Cognitis quantitatum a', a'', a''', &c. valoribus, coëfficientium a, b, c, &c. æquationis (D) valores innotescent per formulas subsequentes ex formulis nuper traditis erutas, exhibendo per A, A'', A''', &c. quantitatum B, C, D, &c. valores, litteris majusculis in minusculas commutatis, nimirum

$a = a'$

$b = \frac{aa' - a''}{2}$

$c = \frac{ba' - aa'' + a'''}{3}$

$d = \frac{ca' - ba'' + aa'''}{4}$

&c. &c.

nam $A' = A = a$.

$A'' = AA' - 2b$; nempe $a'' = aa' - 2b$; hinc $b =$ $\frac{aa' - a''}{2}$.

$A''' = AA'' - BA' + 3C$; nimirum $a''' = aa'' - ba' +$ $3c$; hinc $c = \frac{ba' - aa'' + a'''}{3}$.

A^{IV}

$B^{IV} = AA''' - BA'' + CA' - 4D$; hoc est $a^{IV} = aa''' - ba'' + ca' - 4d$; unde $d = \frac{ca' - ba'' + aa''' - a^{IV}}{4}$. &c.

Inventis igitur per æquationis datæ [A] vel [B] coëfficientes cognitos A, B, C; &c. valoribus quantitatum A', A'', A''', &c. usque ad A, in qua accentuum numerus sit $= 2n$, per istos quærantur valores quantitatum a'; a'', a''', &c. usque ad a, in qua accentuum numerus sit $= n$; in penultima enim formula quantitati a' respondet A'', quantitati a'' respondet A^{IV}, &c. ut illam intuenti palam erit; ac tandem per postremos hosce valores invenientur valores quæsiti coëfficientium a, b, c; &c.

799. *Coroll.* 5. Si in æquatione [D] mutetur x in $-x$, ipsa eadem persistet; quum enim contineat quadrata differentiarum ex radicibus propositæ [B]; perinde erit tam si ponamus, eam constare quadratis $(\alpha-\beta)^2$, $(\alpha-\gamma)^2$, $(\beta-\gamma)^2$, &c. quam si statuamus, ipsam efformari quadratis $(\beta-\alpha)^2$, $(\gamma-\alpha)^2$, $(\gamma-\beta)^2$, &c. mutando igitur signa in locis paribus in æquatione [B], æquatio [D] mutationem non patitur.

800. *Coroll.* 6. Dicta æquatio [D] eadem pariter persistet, si omnes æquationis [B] radices augeantur, vel imminuantur eadem quantitate, quæcumque ea sit; nam $(\alpha \pm \pi) - (\beta \pm \pi) = \alpha - \beta$, &c. adeoque $\overline{(\alpha \pm \pi) - (\beta \pm \pi)}^{\,2} = (\alpha - \beta)^2$, &c. Hinc si æquatio [$B$] secundo termino gaudeat, potest hic eliminari, ac deinde construi, ut diximus, æquatio [D]; quæ eadem permanebit ac si dictus secundus terminus

nus non evanuiſſet. At hujus termini exturbatio id habet commodi, quod cœfficientium a, b, c, &c. perquiſitionem reddit faciliorem, fit enim $A = 0$, ac proinde $A' = 0$; formulæ igitur nuperæ evadent

$A' = 0$	$a' = mA''$	$a = a'$
$A'' = -2B$	$a'' = mA^{IV} + 6\frac{(A'')^2}{2}$	$b = \frac{aa' - a''}{2}$
$A''' = 3C$		
$A^{IV} = -BA'' - 4$	$a''' = mA^{VI} + 15A''A^{IV} - 20\frac{(A''')^2}{2}$	$c = \frac{ba' - a'' + a'''}{3}$
&c.	&c.	&c.

801. *Coroll.* 7. Quoniam æquationis [D] radices ſunt quadrata differentiarum $\alpha - \beta$, $\alpha - \gamma$, $\beta - \gamma$, &c. ſi æquationis hujuſce ſigna ſint omnia ſimilia, ejus radices forent quadrata negativa; ſed quadratum negativum eſt impoſſibile; ergo differentiæ inter radices æquationis [B] forent omnes imaginariæ, velut $t\sqrt{-1}$, vel $-t\sqrt{-1}$.

802. *Coroll.* 8. Si æquatio [B] unum vel plura radicum æqualium paria poſſideat, æquatio [D] unum, vel plures habebit quantitatis v valores $= 0$. Hinc methodus explorandi, an æquatio data habeat nec ne radices invicem æquales; quod ſi tales habeat, illæ ex dictis (514.) eliciantur, & æquatio per debitam diviſionem deprimatur.

803. Propoſitio III. *Radicum æquationis datæ* (A) *vel* (B) *Prop. antec. limites invenire.*

Reſolutio. Poſita $r = n$ ſi æquatio data nullas, vel $r < n$, ſi per diviſionem depreſſa nonnullas prius habuerit radices invicem æquales, æquationis termini invertantur, ita ut ipſa evadat

$$1 + \alpha v + \beta v^2 + \gamma v^3 + \text{\&c.} \ldots\ldots + \pi v^r = 0 \ldots (E).$$

Tum

Tum statue $v = \frac{1}{y}$, & æquatione (E) ordinata per y, habebitur

$$y^r + \alpha y^{r-1} + \beta y^{r-2} + \gamma y^{r-3} + \&c. \ldots + \pi = 0 \ldots (F).$$

Quæratur limes l radicum hac in æquatione degentium, ita ut l superet valores quoscumque positivos incognitæ y (467.). Ergo quoniam $l > y$, erit $\frac{1}{l} < \frac{1}{y}$, seu $\frac{1}{l} < v$, adeoque ob $v = u^2$ (796.), erit $\frac{1}{l}$ minor quovis valore quantitatis u^2, proptereaque $\frac{1}{\sqrt{l}} < u$, nimirum $\frac{1}{\sqrt{l}}$ necessario minor qualibet ex differentiis, quæ intercedunt inter radices reales positivas, & inæquales propositæ (B). Hinc

(1.°) Si denominator $\sqrt{l} = 1$, vel si $\sqrt{l} < 1$, certum erit, nullas in æquatione (B) adesse radices reales, quarum differentiæ sint unitate minores, quia $\frac{1}{\sqrt{l}}$ vel fit $= 1$, vel > 1. Hoc igitur in casu statui procul dubio potest $\Delta = 1$ (792. n. (1.)).

(2.°) Sin autem denominator $\sqrt{l} > 1$, tunc ob fractionem $\frac{1}{\sqrt{l}}$ fieri potest, ut æquatio (B) radices

ha-

habeat, quarum differentiæ fint unitate minores. Sed quum harum differentiarum minima λ, five u, fit semper $> \frac{1}{\sqrt{l}}$; accipi semper poterit $\Delta = \frac{1}{\sqrt{l}}$, vel $\Delta < \frac{1}{\sqrt{l}}$ [*l. cit.*], esse enim debet $\Delta < u$; vel $\Delta = u$.

Generatim si $\sqrt{l}$ extrahi nequeat, quum radicales in substitutionibus locum non habeant, sumatur numerus proxime major quam $\sqrt{l}$, qui fiat $= K$, idemque peragatur, si $\sqrt{l}$ sit numerus integer. Erit in utroque casu $\frac{1}{K} = \frac{1}{\sqrt{l}}$; adeoque sumi semper poterit $\Delta = \frac{1}{K}$, vel $\Delta < \frac{1}{K}$. Invento igitur æquationis (*F*) limite l, sumptoque K proxime majore quam $\sqrt{l}$, fiat $\Delta = \frac{1}{K}$; tum numeris o, $\frac{1}{K}$; $\frac{2}{K}$, $\frac{3}{K}$, &c. pro incognita in æquatione successive suffectis, quæ ex hac operatione prodibunt, seriem formabunt, in qua tot aderunt signorum variationes, quot radices reales positivas & inæquales æquatio comprehendet, & istæ radices inter duos dictæ seriei numeros continuos signis diversis affectos (ex. gr. inter $\frac{b}{K}$, & $\frac{b+1}{K}$) delitescent; quare numerus in-

integer, qui alterutro ex valoribus $\frac{b}{K}$, $\frac{b+1}{K}$ appropinquabit, erit valor integer proximus unius ex radicibus quæsitis. *Quod erat* &c.

804. *Scholion.* Ad fractionum, substitutionumque molestiam imminuendam expedit (1.°) ut pro incognitæ x potestatibus, scribantur in æquatione data potestates fractionis $\frac{x}{K}$, ac deinde per ipsius K maximam potestatem omnes æquationis termini multiplicentur; sic fractiones vitabuntur; (2.°) ut substitutis successive numeris incognitæ $x = 0, 1, 2, 3,$ &c. respondentibus, ex primis quantitatibus $m+1$, quæ sic agendo proveniunt, fiat transitus prius per differentias primas, secundas, tertias, &c. tum per additiones, ad reliquos numeros inveniendos. Exemplo res illustratur.

Esto $x^3 - 63x + 189 = 0$.

Numeri substitutionibus $x = 0, 1, 2, 3$ respondentes erunt 189, 127, 71, 27. Subtrahe 189 ex 127, & residuum -62 superponatur numero 189. Ex 71 deme 127, & residuum -56 collocetur supra 127, & ita porro. Primas hasce differentias excipiant secundæ, subtrahendo -62 ex -56, & residuum 6 scribatur supra -62; pari modo subducto -56 ex -44, residuum 12 notetur supra -56. Eodem ordine deveniatur ad tertiam differentiam, subtrahendo 6 ex 12, & ponendo 6 supra 6. Quibus

bus sic dispositis, ut operatio absolvatur, quatuor series subsequentes

6	6	6	6	6	6	6	&c.
6	12	18	24	30	36	42	&c.
— 62	— 56	— 44	— 26	— 2	28	64	&c.
189	127	71	27	1	— 1	27	&c.

efficientur, quarum ea est lex, ut quilibet terminus adæquet aggregatum ex termino antecedente ejusdem seriei, & ex illo, qui ipsi superponitur, ut videre est in paradigmate. Sic $27 - 26 = 1, - 26 + 24 = - 2$, &c. quæ series continuari pro libito possunt; ubi advertendum, ultimam ex quatuor hisce seriebus prodire ex substitutionibus numerorum naturalium 0, 1, 2, 3, &c. pro x in proposita factis. Quum autem omnes septimæ columnæ verticalis termini, videlicet 6, 42, 64, 27 sint positivi, liquet, terminos omnes subsequentes fore pariter positivos, ita ut si numerorum ex substitutionibus emergentium seriem continuare libeat, nulla amplius signorum occurret variatio.

CAPUT VII.

DE RADICIBUS IMAGINARIIS.

805. Lemma. *Quantitatum realium* $\alpha, \beta, \gamma, \delta$, *&c. quarum numerus* $= m$; *summa ponatur* $= M$; *quadratorum* $\alpha^2, \beta^2, \gamma^2, \delta^2$, *&c. summa* $= N$; *productorum ex binis* $\alpha\beta, \alpha\gamma, \alpha\delta, \beta\gamma, \beta\delta$, *&c. summa* $= L$; *dico, fore semper* $\frac{(m-1)}{2} N > L$.

Dem.

Dem, Sumptis omnibus datarum quantitatum differentiis $\alpha - \beta, \alpha - \gamma, \alpha - \delta, \beta - \gamma$, &c. liquet, in hujusmodi differentiarum quadratis sumi α^2 tot vicibus, quot sunt datæ quantitates una minus, quod valet quoque de β^2, γ^2, &c. rectangula vero $-2\alpha\beta, -2\alpha\gamma, -2\alpha\delta, -2\beta\delta$, &c. semel tantum efficiuntur; quare summa omnium quadratorum $(\alpha - \beta)^2 + (\alpha - \gamma)^2 + (\alpha - \delta)^2 + (\beta - \gamma)^2 +$ &c. $= (m-1)(\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \&c.) - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\delta + \&c.) = (m-1)N - 2L$. Sed $(\alpha - \beta)^2 + (\alpha - \gamma)^2 + (\alpha - \delta)^2 +$ &c. est semper quantitas positiva, quod omnia quadrata sunt positiva; Ergo $(m-1)N - 2L$ est quantitas positiva, adeoque $\frac{(m-1)}{2}N > L$. *Quod erat* &c.

806. *Coroll.* Quoniam $L = (\alpha + \beta + \gamma + \delta + \&c.)^2 - (\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \&c.) = \frac{MM - N}{2}$; erit

$(m-1)N - \frac{2(MM - N)}{2} = mN - MM$, ac proinde $mN > MM$.

Scholion. Quamvis quantitates $\alpha, \beta, \gamma, \delta$, &c. adsumptæ sint positivæ, attamen quæ demonstravimus vera quoque sunt, si datæ quantitates ponantur negativæ; earum siquidem quadrata eadem persistunt, productorum vero summa vel est eadem, vel minor quam si fuerint positivæ.

807. Propositio I. *Data quavis æquatione radices omnes reales habente, cujuslibet coëfficientis quadratum*

tum non solum cum producto coëfficientium proxime adjacentium, sed cum rectangulis quoque binorum quorumcumque utrinque aeque distantium comparare, hac tamen conditione, ut hujusmodi rectangulorum signa alternatim mutentur.

Resolutio. Esto $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \&c. = 0$, cujus aequationis radices reales numero n sint a, b, c, d, e, &c. Ex aequationum natura terminorum coëfficientes, cum terminorum numero e regione stabunt (396.).

$A = a + b + c + d + \&c. \ldots\ldots$	n
$B = ab + ac + ad + bc + bd + \&c. \ldots\ldots$	$\frac{n(n-1)}{1.\ 2}$
$C = abc + abd + abe + acd + bcd + \&c.$	$\frac{n(n-1)(n-2)}{1.\ 2.\ 3}$
$D = abcd + abce + abde + \&c. \ldots\ldots$	$\frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4}$
&c.	&c.

Singularum hujusmodi serierum quadrata sequenti modo constituantur, nimirum

$P = a^2 + b^2 + c^2 + d^2 + \&c.$

$Q = a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + \&c.$

$R = a^2b^2c^2 + a^2b^2d^2 + a^2b^2e^2 + a^2c^2d^2 + \&c.$

$S = a^2b^2c^2d^2 + a^2b^2c^2e^2 + a^2b^2d^2e^2 + \&c.$

&c.

Quum autem sit $a^2 + b^2 + c^2 + \&c. = (a + b + c + \&c.)^2 - 2(ab + ac + ad + \&c.)$, erit $P = A^2 - 2B$. Similiter ob $a^2b^2 + a^2c^2 + a^2d^2 + \&c. = (ab + ac + ad + \&c.)^2$ —

$-2(a^2bc + a^2bd + a^2be + a^2cd +$ &c.$) + 2(abcd + abce + abde +$ &c.$)$ ſtabit $Q = B^2 - 2AC + 2D$; quare ſic proſequendo, habetur

$P = A^2 - 2B$

$Q = B^2 - 2AC + 2D$

$R = C^2 - 2BD + 2AE - 2F$

$S = D^2 - 2CE + 2BF - 2AG + 2H$

&c.

Per Lemmatis igitur Corollarium quum P locum teneat ipſius N, & A ipſius M, ſi in primo caſu pro m ſumatur n, erit $nP > AA$. In ſecundo caſu quoniam Q ſtat pro N, & B pro M, ſumpto terminorum numero $\frac{n(n-1)}{1.\ 2}$ pro m, fit $\frac{n(n-1)}{1.\ 2}Q > BD$, & ſic deinceps; ex quo prodit

$nP > AA$

$\frac{n(n-1)}{1.\ 2}Q > BB$

$\frac{n(n-1)(n-2)}{1.\ 2.\ 3}R > CC$

$\frac{n(n-1)(n-2)(n-3)}{1.\ 2.\ 3.\ 4}S > DD.$

&c.

ſuffice pro P, Q, R, &c. valores nuper expoſitos, & obtinebis $nAA - 2nB > AA$, ſive $AA > \frac{2n}{n-1}B$.

$\frac{n(n-1)}{1 \cdot 2} BB - \frac{2n(n-1)}{1 \cdot 2} AC + \frac{2n(n-1)}{1 \cdot 2} D > BB$, five

$$BB > \frac{\frac{2n(n-1)}{1 \cdot 2}}{\frac{n(n-1)}{1 \cdot 2} - 1}(AC - D)$$; ac fimiliter agendo

$$CC > \frac{\frac{2n(n-1)(n-2)}{1 \cdot 2 \cdot 3}}{\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} - 1}(BD - AE + F)$$

$$DD > \frac{\frac{2n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}}{\frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} - 1}(CE - BF + AG - H)$$

&c.

Quod erat &c.

808. *Coroll.* Ergo in binomio ad poteſtatem ejuſdem gradus cum æquatione propoſita elevato accipiantur unciæ terminorum, quæ duplicatæ ordinatim ſuperſcribantur terminis ejuſdem æquationis, primo & ultimo exceptis; deinde quælibet uncia dividatur per ſeipſam unitate imminutam. Quod ſi quadratum cujuslibet termini per ſupereminentem fractionem multiplicatum majus ſit rectangulo ex duobus primis terminis proxime adjacentibus — rectangulo ex duobus ſecundis adjacentibus + rectangulo ex duobus tertiis æquidiſtantibus, & ſic deinceps, tunc iſti termino ſubſcribatur ſignum +, ſecus —, ſtante ſemper ſigno + ſub primo, & ultimo termino,

no, operatione in singulis æquationis terminis simili modo absoluta, & signorum ordine rimato, quot reperientur signorum mutationes, tot indicabuntur radices imaginariæ.

Exemplum. Esto $x^7 - 5x^6 + 15x^5 - 23x^4 + 18x^3 + 10x^2 - 28x + 24 = 0$. Unciæ binomii ad septimam potestatem elati sunt 7, 21, 35, 35, 21, 7. Ergo æquatio cum fractionibus supereminentibus evadet

$$x^7 - \overset{\frac{7}{3}}{5x^6} + \overset{\frac{21}{10}}{15x^5} - \overset{\frac{35}{17}}{23x^4} + \overset{\frac{35}{17}}{18x^3} + \overset{\frac{21}{10}}{10xx} - \overset{\frac{7}{3}}{28x} + 24 = 0$$

$$+ \quad + \quad - \quad + \quad - \quad + \quad - \quad +$$

Etenim collocato signo + sub terminis extremis x^7, & 24, quia quadratum secundi termini ductum in fractionem supereminentem $\frac{7}{3}$ exhibet $\frac{175}{3}$, quod factum majus est rectangulo ex adjacentibus $\frac{315}{10}$, ponatur sub secundo termino signum +. Quia vero quadratum tertii termini ductum in supereminentem fractionem $\frac{21}{10}$ minus est quam rectangulum ex secundo & quarto, imminutum rectangulo ex primo & quinto, notetur signum —. Præterea ob quadratum quarti termini multiplicatum per fractionem supereminentem majus rectangulo ex primis terminis adjacentibus — rectangulo ex secundis + rectangulo ex tertiis, ponatur signum +; & sic prosequendo, omnia signa exarabis, in quibus sex

animadvertes mutationes, quæ sex radices imaginarias in proposita indicabunt; profecto istæ sunt $1 + \sqrt{-3}$, $1 - \sqrt{-3}$; $1 + \sqrt{-2}$, $1 - \sqrt{-2}$; $1 + \sqrt{-1}$, $1 - \sqrt{-1}$.

809. *Scholion.* Hanc Regulam, quæ extat in additamento ad *Newtoni* Arithmeticam universalem, protulit cel. *Georgius Campbell*, eamque illa exacti rem ostendit, quam summus *Newtonus* in dicta exhibuit arithmetica. Attamen fallax & ipsa Campbelliana sæpius reperitur, signorum enim distributiones ad normam corollarii præcedentis institutæ nullas indicare aliquando possunt radices imaginarias, quamvis ipsas æquatio contineat. Æquationes ipsæ cubicæ, & quidem infinitæ formari possunt hujusmodi prædictæ proprietate. Generatim vero id evenit in æquationibus, ubi bina signa consimilia se mutuo excipiunt. Tales sunt

$$x^n - Ax^{n-1} - Bx^{n-2} + Cx^{n-3} + Dx^{n-4} - Ex^{n-5} - Fx^{n-6} + \&c. = 0.$$

$$x^n + Ax^{n-1} + Bx^{n-2} - Cx^{n-3} - Dx^{n-4} + Ex^{n-5} + Fx^{n-6} - \&c. = 0$$

Sic æquatio biquadratica $x^4 + 4x^3 - 8x^2 - 24x + 108 = 0$ omnes habet radices imaginarias, quippe quæ resolvitur in duos factores quadraticos

$$xx + 8x + 18 = 0, \& \; xx - 4x + 6 = 0,$$

etsi nullum earum indicium per expositam Regulam appareat. Hac de re consulendus est celeberrimus

mus Eulerus tam in sui Calculi Differentialis Parte II. quam in novis Monumentis Acad. Scient. Imper. Petrop. Tom. XIII. pro an. 1768. pag. 89. seq. Quum itaque non semper certa sit methodus Campbelliana, aliam quamvis inversam in fine capitis sequentis exponam.

810. Propositio II. *Radices imaginarias in æquatione proposita latentes in apricum proferre.*

Resolutio. Sint e, f, g, h, &c. radices reales, & $\alpha + \beta\sqrt{-1}$, $\alpha - \beta\sqrt{-1}$, $\gamma + \delta\sqrt{-1}$, $\gamma - \delta\sqrt{-1}$, &c. radices imaginariæ; Duo casus dari possunt, nimirum vel nulla ex realibus e, f, g, h, &c. nullam ex realibus α, γ, &c. adæquat, vel aliqua ex primis realibus alicui ex secundis realibus est æqualis.

Quoad primum, æquationum in Capite præcedenti expositarum retentis designationibus (B), (C), (D), &c. ut proposita (B) radicibus omnibus realibus gaudeat, incognitæ u in æquatione (C) valores esse pariter debent reales; ex quo sequitur, valores incognitæ u^2, vel v in æquatione (D) esse omnes positivos, adeoque omnes ipsius radices reales, & positivas, seu, quod eodem recidit, ipsius signa per vices procedere (441.). Ergo si hujusmodi conditionibus careat æquatio (D), seu si aliquam habeat signorum consecutionem, certum erit hoc in casu æquationem (B) radices imaginarias comprehendere. Jam vero radices imaginariæ, quæ binæ semper adsunt, generatim repræsentantur sub forma $\alpha + \beta\sqrt{-1}$, $\alpha - \beta\sqrt{-1}$ (440.), quare ipsarum differentia est $\pm 2\beta\sqrt{-1} = u$, ac proinde $u^2 = v = -4\beta^2$. Ergo quot radices reales negativas habebit æqua-

tio (Q), tot radicum imaginariarum paria in propoſita (B) neceſſario deliteſcent. Quare ſi ponatur $v = - w$, æquatio (D) migrabit in novam æquationem

$$w^n - aw^{n-1} + bw^{n-2} - cw^{n-3} + \&c. = 0 \ldots\ldots (G),$$

quæ tot radicibus realibus poſitivis conſtabit, quot radicum imaginariarum paria æquatio (D) continebit, adeo ut ſi unicam ſignorum variationem æquatio (G) præſeferat, duas tantum propoſita (B) radices imaginarias includet. Hiſce præmonitis, finge, allatæ æquationis (G) radices eſſe w, w', w'', w''', &c. Ob $v = - w = - 4\beta^2$, valores β erunt $\frac{\sqrt{w}}{2}$, $\frac{\sqrt{w'}}{2}$, $\frac{\sqrt{w''}}{2}$, $\frac{\sqrt{w'''}}{2}$, &c. Quum autem valores α remaneant dignoſcendi; in æquatione (B) pro x ſubſtitue $\alpha + \beta\sqrt{-1}$, ſed ita ut ſeparatis terminis omnibus realibus ab imaginariis, exurgant duæ ſeries ordinatæ per α, quæ ſingulæ poſitæ $= 0$, & idcirco deletis in harum altera expreſſionibus $\sqrt{-1}$; erunt hujus formæ

$$\left.\begin{array}{l} \alpha^m + P\alpha^{m-1} + Q\alpha^{m-2} + \&c. = 0 \\ m\alpha^{m-1} + p\alpha^{m-2} + q\alpha^{m-3} + \&c. = 0 \end{array}\right\} \ldots\ldots [H]$$

ubi ob dictam ſubſtitutionem coëfficientes P, Q, &c. p, q, &c. dantur per A, B, C, &c. [793.], & per β. Suffectis igitur pro β valoribus prænotatis, ob eandem incognitam α utraſque has noviſſimas æquationes $[H]$ uno gradu diverſas ingredientem, patet, iſtas

istas communi divisore præditas esse. Inveniatur igitur maximus communis divisor, donec perventum sit ad residuum, in quo quantitas α ad primam dimensionem ascendat, quo residuo posito $=0$, æquatio proveniet quantitatibus α, & β affecta, in qua quum β innotescat, innotescet etiam α. Quod si inter valores ipsius β ex æquatione [G] erutos duo adsint invicem æquales, seu si β bis occurrat ducta in $\sqrt{-1}$, tunc quoniam singulis hisce valoribus duo diversi valores quantitatis α respondere possunt, duplex hic ejusdem β valor in æquationes [H] necessario transibit, adeoque duabus radicibus communibus æquationes istæ conslabunt, ac proinde divisio pro maximo earum communi divisore venando continuari debet usque ad residuum, in quo α ad duas dimensiones elata remaneat, quo ut ante posito $=0$, ex duabus ejus radicibus duo quantitatis α valores ambo reales educentur. Similiter in dicto residuo quantitas α ad tertium gradum debet assurgere si β tres habeat valores invicem æquales, & sic deinceps. *Quod erat primum.*

Quoad secundum, esto $e=\alpha$; Quadrata differentiarum $e-\alpha-\beta\sqrt{-1}$, $e-\alpha+\beta\sqrt{-1}$, quæ sunt æquationis [D] radices, evadent $-\beta^2$, $-\beta^2$; quare si æquatio [B] duas tantum contineat radices imaginarias $\alpha+\beta\sqrt{-1}$, $\alpha-\beta\sqrt{-1}$, tunc æquatio [D], præter radicem negativam $-4\beta^2$, duas quoque $-\beta^2$, $-\beta^2$ invicem æquales comprehendet. Ergo si dicta æquatio [D] tres habeat radices reales negativas, quarum duæ sint invicem æquales, proposita [B] tria continere potest ex nuper dictis radicum imagina-

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riarum paria, vel unum tantummodo par per præsentem casum. Quod si proposita [B] quatuor habeat radices imaginarias $\alpha+\beta\sqrt{-1}$, $\alpha-\beta\sqrt{-1}$; $\gamma+\delta\sqrt{-1}$, $\gamma-\delta\sqrt{-1}$, tunc æquatio (D), præter duas radices reales negativas $-4\beta^2$, $-4\delta^2$, duas insuper habebit $-\beta^2$, $-\beta^2$, si $e=\alpha$; nec non $-\delta^2$, $-\delta^2$, si $f=\gamma$. Si demum fuerit $\alpha=\gamma$, sumptis differentiis, & in compendium redactis, quatuor ejusdem æquationis (D) radices imaginariæ
$(\alpha-\gamma+(\beta-\delta)\sqrt{-1})^2$, $(\alpha-\gamma-(\beta-\delta)\sqrt{-1})^2$
$(\alpha-\gamma+(\beta+\delta)\sqrt{-1})^2$, $(\alpha-\gamma-(\beta+\delta)\sqrt{-1})^2$
evaderent $-(\beta-\delta)^2$, $-(\beta-\delta)^2$, $-(\beta+\delta)^2$, $-(\beta+\delta)^2$, reales nempe negativæ, nec non binæ invicem æquales. Hinc colligitur, quod si inter radices reales negativas æquationis [D] nonnullæ existant invicem æquales, tunc quævis ex radicibus æqualibus radicum imaginariarum par unum, velut in primo casu, exhibebit; at quodlibet radicum æqualium par vel duo radicum imaginariarum paria, vel nulla suppeditabit. Sic duæ radices æquales vel quatuor radices imaginarias, vel nullas præbebunt; Tres radices æquales vel sex vel duas; Quatuor radices æquales vel octo, vel quatuor radices imaginarias indicabunt; atque ita porro.

Ponamus enim, duas radices æquales æquationis [D] esse $-w$, $-w$; Fiat, ut supra, $\beta=\frac{\sqrt{w}}{2}$, quo valore in æquationibus [H] suffecto, quæratur communis earundem divisor, ita ut deveniatur ad residuum, in quo α ad duas dimensiones (ob dupli-

plicem ipſius β valorem) aſcendat; hoc reſiduo poſito $= 0$, duæ radices reales, vel duæ imaginariæ pro valore α determinando prodibunt; quapropter ſi ejuſdem α valores emergant imaginarii, quum id contra hypotheſim eveniat, duæ radices æquales $-w$, $-w$ nullas indicabunt adeſſe in propoſita radices imaginarias. Similiter ſi eadem æquatio $[D]$ tres includat radices reales negativas $-w, -w, -w$, ponendo de more $\beta = \frac{\sqrt{w}}{2}$, dicta æquationum $[H]$ diviſio continuetur, donec perveniatur ad reſiduum, ubi (ob triplicem quantitatis β valorem) α ſit ad tertiam poteſtatem evecta; quo reſiduo facto $= 0$, habebitur æquatio tertii gradus, ubi α vel tres exhibebit valores reales, vel unum realem ac duos imaginarios; ſi primum, ſex æquatio data continebit radices imaginarias; ſi alterum, duas tantum, rejectis imaginariis hypotheſi adverſantibus; & ſic deinceps; *Quod erat ſecundum.*

Exemplum I. Eſto $x^3 * + x + 10 = 0$. Quoniam $\frac{m[m-1]}{2} = n = 3$, $A = 0$, $B = 1$, $C = 10$, erit

$A'' = -2B = -2$; $A''' = 3C = 30$; $A^{IV} = -BA'' = 2$, ob $D = 0$;

$A^{V} = -BA''' + CA'' = -50$; $A^{VI} = -BA^{IV} + CA''' = 298$. Quare

$a' = nA'' = -6$

$a'' = mA^{IV} + 3[A'']^2 = 18$

$a'' = mA^{vi} + 15A''A^{iv} - 10[A''']^2 = -8166$. Hinc $a = -6$; $b = 9$; $c = -2704$.
Subſtitue hos valores in æquatione [G], & habebis $w^3 - 6w^2 + 9w - 2704 = 0$, cujus radix realis poſitiva $= 16$. Ergo $\frac{\sqrt{w}}{2} = \frac{\sqrt{16}}{2} = 2 = \beta$.

Ut inveniatur α, pro x in propoſita $x^3 + x + 10 = 0$ ſuffice $\alpha + \beta\sqrt{-1}$, ſive $\alpha + 2\sqrt{-1}$; ex quo fit
$\alpha^3 + 6\alpha^2\sqrt{-1} - 12\alpha - 8\sqrt{-1} + \alpha + 2\sqrt{-1} + 10 = 0$;
unde duæ obtinentur æquationes $\alpha^3 - 11\alpha + 10 = 0$, & $\alpha - 1 = 0$, ubi eſt inutile primam per ſecundam æquationem dividere uſque ad reſiduum, in quo α primam teneat dimenſionem, ſtatim enim conſpicitur $\alpha = 1$. Propoſitæ igitur radices imaginariæ ſunt $1 \pm 2\sqrt{-1}$, quod per examen verum reperietur.

Exemplum II. Eſto $x^3 - 8x^2 + 25x - 26 = 0$, ubi $A = -8$, $B = 25$, $C = -26$. Determinatis igitur ut ſupra coëfficientibus a, b, c, factiſque ſolitis ſubſtitutionibus, proveniet $w^3 - 22w^2 + 121w - 400 = 0$, cujus radix realis poſitiva $= 16$, quamobrem $\frac{\sqrt{16}}{2} = 2 = \beta$. Ut obtineatur valor α, ſubroga $\alpha + \beta\sqrt{-1}$, ſive $\alpha + 2\sqrt{-1}$ in propoſita, & invenies $\alpha^3 + 6\alpha^2\sqrt{-1} - 12\alpha - 8\sqrt{-1} - 8\alpha^2 - 32\alpha\sqrt{-1} + 32 + 25\alpha + 50\sqrt{-1} - 26$, unde duæ ſecernuntur æquationes
$\alpha^3 - 8\alpha^2 + 13\alpha + 6 = 0$
$3\alpha^2 - 16\alpha + 21 = 0$.

Mul-

Multiplica primam æquationem per 3, ut ejus inchoari possit per secundam divisio [29.], qua peracta, primum residuum apparebit $-8x^2+18x+18$, quod in eundem finem multiplica per 3, ac deinde divide per secundam æquationem pro secundo residuo, in quo x sit unius dimensionis; habebis $-74x+222=0$, sive $x=3$; ergo radices imaginariæ quæsitæ sunt $3\pm2\sqrt{-1}$.

CAPUT VIII.

DE ÆQUATIONUM PER APPROXIMATIONEM RESOLUTIONE.

811. Propositio I. *Ex æquationibus puris cujuscumque gradus radicem per approximationem extrahere.*

Resolutio. *Esto* $x^n=a$. Ut radicem $\sqrt[n]{a}$ inveniam, pono, hanc esse $=c$ proxime, ita ut sit $x=c+g$. Efferatur $c+g$ ad potestatem n, & in seriem resolvatur [67.]. Quoniam vero g est semper fractio, fient g^2, g^3 [& multo magis elatiores ipsius fractionis g potestates] unitate minores; quare quum de approximatione sermo sit, earum supputationem omittere tuto possumus. Habebimus igitur $c^n+nc^{n-1}g$, adeoque $x^n=c^n+nc^{n-1}g=a$; hinc $nc^{n-1}g=a-c^n$, & $g=\frac{a-c^n}{nc^{n-1}}$; ac tandem $x=$

$c+$

$c+g = \frac{nc^n - c^n + a}{nc^{n-1}} = \frac{[n-1]c^n + a}{nc^{n-1}}$. Quum itaque posito valore c proxime minore quam radix quæsita, detur etiam g, adeoque x innotescat, iste, quippe qui exprimit radicem proximam quæsitam, accipi rursus potest pro valore c, ita ut statuendo $x = c+g$, ad exactiorem ipsius x valorem accedere, si libeat, valeamus, & ita porro. *Quod erat* &c.

Exemplum I. Esto $x^2 = 2$. Erit $n = 2$, $a = 2$, $c > 1$, sed < 2; ergo $x = 1 + g = \frac{1+2}{2} = \frac{3}{2}$.

Fiat $c = \frac{3}{2}$; erit $x = \frac{3}{2} + g = \left(\frac{9}{4} + 2\right) : 2 \cdot \frac{2}{3} = \frac{17}{12}$. Ponatur $c = \frac{17}{12}$. Erit $x = \frac{577}{408}$; qui valor ita est proximus $\sqrt{2}$, ut ejus quadratum, $\frac{332929}{166464}$ a numero 2 non differat, nisi fractione $\frac{1}{166464}$, qua ipsum superat; si enim statuas $2 = \frac{332929}{166464}$, ac deinde multiplices per 166464, prodibit ex una parte 332928, ex altera 332929. Quod

Quod si libeat ulterius progredi, fac $c = \frac{577}{408}$, & invenies $x = \frac{271669656}{192099456}$.

Exemplum II. Esto $x^3 = 2$. Ut obtineas $\sqrt[3]{2}$, pone $c = 1$, $n = 3$, $a = 2$; erit $x = \frac{4}{3}$. Sume $c = \frac{4}{3}$; Erit $x = \frac{91}{72}$. Fac $c = \frac{91}{72}$; erit $x = \frac{162130896}{128634294}$.

Aliter. Esto $a + f$ radix gradus n, quæ jam extracta ex $a^n \pm b$ concipiatur, & a exprimat numerum integrum, f fractionem decimalem. Erit igitur $a + f = \sqrt[n]{a^n \pm b}$. Hinc $[a+f]^n = a^n \pm b = a^n + na^{n-1}f + \frac{n[n-1]}{2}a^{n-2}f^2 + \ldots$ &c. quapropter si terminis, ubi fractio f ad potestates quadrato majores assurgit, prætermissis, ac deleta utrinque a^n, reliquum dividatur per n, fiet $\pm\frac{b}{n} = a^{n-1}f + \frac{n-1}{2}a^{n-2}f$; seu multiplicando per 2, & dividendo per $[n-1]a^{n-2}$, & æquationem, ordinando,

do, $f^2 + \frac{2a}{n-1} f = \pm \frac{2b}{[n^2 - n] a^{n-2}}$; ex quo habetur $f = -\frac{a}{n-1} \pm \sqrt{\frac{a^2}{[n-1]^2} \pm \frac{2b}{[n^2 - n] a^{n-2}}}$. Itaque addita hinc inde a, prodibit $a + f = \sqrt[n]{a^n \pm b} = \frac{n-2}{n-1} a + \sqrt{\frac{a}{[n-1]^2} \pm \frac{2b}{[n^2 - n] a^{n-2}}}$. *Quod erat* &c.

812. *Coroll.* Ergo

Si $n = 3$, erit $\sqrt[3]{a^3 \pm b} = \frac{1}{2} a + \sqrt{\frac{1}{4} aa \pm \frac{b}{3a}}$

Si $n = 4$, erit $\sqrt[4]{a^4 \pm b} = \frac{2}{3} a + \sqrt{\frac{1}{9} aa \pm \frac{b}{6aa}}$

Si $n = 5$, erit $\sqrt[5]{a^5 \pm b} = \frac{3}{4} a + \sqrt{\frac{1}{16} aa \pm \frac{b}{10a^3}}$; &c.

Ubi proximus quantitatis $a + f$ valor, quicumque sit numerus n, ad secundæ tantum radicis extractionem reducitur.

Exemplum. Elicienda esto radix quinta ex 161900 cum 12 decimalibus. Logarithmum numeri 161900, qui est 5, 2092468, divide per 5; quotus erit 1, 0418494, logarithmus nempe numeri 11, 012, quæ est radix proxima. Pone 11, 012 $= a$; attolle 11, 012 ad quintam potestatem, & habebis $a^5 = 161931, 37873202072883 2$, cujus excessus supra

pra 161900 eſt 31, 37873202072883 2. Hunc exceſſum ſtatue $=b$; fiet $a^5 - b = 161900$. Ergo per formulam $\sqrt[5]{a^5-b} = \frac{3}{4}a + \sqrt{\frac{1}{16}aa - \frac{b}{10a^3}}$, factis

numerorum ſubſtitutionibus, habebis $\sqrt[5]{a^5-b} = 8,259$

$$+\sqrt{\left(7,579009 \frac{31,378732020728832}{13353,60753728}\right)} = 8,259$$

$$+\sqrt{(7,579009 - 0,00234983182882431593 2711)}$$
$$= 8,259 + \sqrt{[7,576659168171175684067289]}$$
$$= 8,259 + 2,752573190339 = 11,011573190339;$$

quæ eſt radix quæſita.

813. *Scholion.* Uſus logarithmorum in radicibus extrahendis eſt inſignis, ſed accidere aliquando poteſt, ut oporteat habere radicem cum multo decimalium numero, quod quum logarithmi vulgares præſtare nequeant, ad formulas ſupra traditas eſt confugiendum.

814. Propoſitio II. *Ad verum radicum æquationis cujuſcumque valorem quantum libuerit accedere.*

Reſolutio. Eſto $Ax^m + Bx^{m-1} + Cx^{m-2} +$ &c. $+ K = 0 \ldots [a]$ radices omnes reales & poſitivas continens, & unius ex iſtis valor integer proximus ponatur per traditam methodum [803.] jam inventus, qui dicatur p, adeo ut ſit $x > p$, & $x < p+1$. Fiat $x = p + \frac{1}{y}$, tum hoc valore pro incognita x

in

in æquatione propoſita ſuffecto, multiplicetur transformata per y^m, & omnes termini ordinentur per y; proveniet $A'y^m + B'y^{m-1} + C'y^{m-2} +$ &c. $+ K' = 0 \ldots [b]$.

Quum autem ſit $\frac{1}{y} > 0$, nec non < 1, erit $y > 1$; Æquatio igitur (b) una ſaltem radice reali majore quam unitas conſtabit; ac proinde radicis hujuſce valor integer proximus, ut jam monuimus [*l. cit.*] indagetur, incognitam y velut radicem poſitivam conſiderando.

Valore integro proximo ipſius u invento, qui vocetur q, ſtatuatur $y = q + \frac{1}{z}$, quo valore pro y in æquatione $[b]$ ſubrogato, tertia æquatio ordinata per z ſic prodibit

$A''z^m + B''z^{m-1} + C''z^{m-2} +$ &c. $+ K'' = 0 \ldots [c]$ atque iſta per ratiocinium ſuperius unam ſaltem neceſſario habebit radicem realem unitate majorem, cujus valor integer proximus inveniri pariter poterit. Hoc proximo radicis z valore vocato r, ponatur $z = r + \frac{1}{u}$; quare ſubſtituendo ut ſupra, emerget æquatio per u, unam ſaltem radicem realem continens unitate majorem; qua ſucceſſiva operatione pro libito continuanda ad verum radicis ſub initium

tium quæsitæ valorem major semper fiet approximatio. *Quod erat* &c.

815. *Coroll.* 1. Si aliqua ex quantitatibus p, q, &c. sit radix exacta, tunc habebitur $x = p$, vel $y = q$, &c. & operatio cessabit, tunc enim incognitæ x valor erit rationalis, in ceteris vero casibus radicis valor erit irrationalis, ad quem tam prope accedere possumus quantum libuerit.

816. *Coroll.* 2. Si pluribus radicibus realibus positivis proposita gaudeat, valor integer proximus cujuslibet ex hisce radicibus per methodos expositas (814.) reperietur, nimirum si valores isti, qui vocentur p, p', p'', &c. successive pro incognita in æquatione substituantur, ut ad verum cujuscumque radicis valorem semper propius accedamus; qua occasione nonnulla sunt animadvertenda, ut sequentia docent.

817. Propositio III. *Si radicum limites successive pro incognita in æquatione a radicibus invicem æqualibus libera substituendi, inter se diversi ponantur, æquationes* (b), (c) *una tantum radice reali majore quam unitas constabunt. Si duo, vel plures dentur limites invicem æquales, æquationes* (b), (c) *totidem radicibus realibus unitate majoribus gaudebunt; & sic de reliquis æquationibus subsequentibus.*

Dem. Quoniam in primo casu positum fuit, esse $x = p + \frac{1}{y}$, si y duos habeat valores diversos, puta m, n, etiam x duos diversos valores habebit, ni-

nimirum erit $x = p + \frac{1}{m}$, & $x = p + \frac{1}{n}$, ad quos idem limes vel valor integer appropinquans p pertinebit; sed limites omnes esse debent (*ex hyp.*) diversi; ergo incognitæ x valor unicus erit $p + \frac{1}{y}$, adeoque unicus in æquatione (b) erit valor realis radicis y unitate majoris. Idem de succedentibus æquationibus (c), (d) &c. in pari casu statuendum.

Quod si ob plures incognitæ x valores, quorum quantitas integra appropinquans p eadem sit, fiat $x = p + \frac{1}{m}$, $x = p + \frac{1}{n}$, &c. adeoque adsumpta y plures habeat valores m, n, &c. tunc quia y æquationis (b) radices repræsentat, æquatio (b) totidem radices reales, singulas unitate majores continebit. Idem de succedentibus æquationibus (c), (d), &c. in pari casu statuendum. *Quod erat* &c.

818. *Coroll.* 1. Ergo pro inveniendis valoribus integris appropinquantibus q, r, &c. radicum in æquationibus $[b]$, $[c]$ existentium, sufficit in primo casu pro y, z, &c. successive substituere numeros naturales positivos 1, 2, 3, &c. donec quidquid oritur ex duorum numerorum continuorum substitutionibus duo signa contraria exhibeat, tunc enim inter duos hujusmodi numeros se statim subsequentes radix delitescet [429. 792.].

819.

819. *Coroll.* 2. Si x duos habeat valores integros appropinquantes p, quoniam hoc valore adhibito singulæ æquationes [b], [c] duas habent radices reales unitate majores, continuetur operatio, donec ad æquationem deveniatur, cujus duæ radices unitate majores duos habeant valores integros appropinquantes diversos; tunc unusquisque ex duobus hisce valoribus peculiarem æquationum seriem præbebit, quarum quælibet unam tantum radicem realem unitate majorem possidebit. Similiter est agendum, si æquatio [a] tres vel plures habeat radices reales positivas, quæ eodem valore integro appropinquante præditæ sint.

820. *Scholion.* Inventis radicibus realibus positivis, pro negativis adsequendis, substitui, ut monuimus [443.], in proposita debet $-x$ in locum x, & novæ hinc prodeuntis æquationis quærantur de more radices reales positivæ, quæ erunt radices negativæ æquationis propositæ. Quod attinet ad radices imaginarias, quæ sub forma $\alpha + \beta\sqrt{-1}$ semper exprimuntur, exhibuimus in Capite antecedente æquationes [H], quarum α, β sunt radices reales. Hisce igitur repertis, radicum omnium imaginariarum in proposita latentium valores obtinebuntur.

821. Propositio IV. *Radicum valores quamproximos per fractiones continuas prius exprimere, deinde ad minimas fractiones communes reducere.*

Resolutio. Sint p, q, r, s, t, &c. valores integri proximi æquationum [a], [b], [c], &c. ita ut habeatur $x = p + \frac{1}{y}$, $y = q + \frac{1}{z}$, $z = r + \frac{1}{u}$, &c.

Hisce valoribus pro incognitæ x valore substitutis, fiet

$$x = p + \cfrac{1}{q + \cfrac{1}{r + \cfrac{1}{s + \&c.}}}$$

Quod erat primum.

Ponatur $\alpha = p$ | $\alpha' = 1$
$\beta = q\alpha + 1$ | $\beta' = q\alpha' = q$
$\gamma = r\beta + \alpha$ | $\gamma' = r\beta' + \alpha'$
$\delta = s\gamma + \beta$ | $\delta' = s\gamma' + \beta'$
&c. | &c.

Hinc pro fractionibus continuis $\frac{p}{1}$, $p + \frac{1}{q}$, $p + \cfrac{1}{q + \cfrac{1}{r}}$, &c. in fractiones communes redigendis, hæc fractionum particularium profluit series

$$\frac{\alpha}{\alpha'}, \frac{\beta}{\beta'}, \frac{\gamma}{\gamma'}, \frac{\delta}{\delta'}, \text{ \&c. } (769.)$$

Istas enim fractiones, scimus, esse convergentes versus incognitæ x valorem; itidemque primam hoc valore minorem, secundam majorem, tertiam minorem, & sic deinceps (773.), adeo ut valor quæsitus inter duas quascumque hujusmodi fractiones sese statim subsequentes semper reperiatur. *Quod erat secundum.*

Exemplum I. Esto $x^3 - 2x - 5 = 0$.

Revocatis denominationibus æquationum sub initium Cap. VI. hujus sectionis exhibitarum, æquationis

(*D*) coëfficientes a, b, c, &c. sunt primum per formulas traditas (800.) inquirendi. Itaque (respice æquationem (*A*), vel (*B*)) statuendo $m = 3$, $A = 0$, $B = -2$, $-C = -5$, seu $C = 5$, habetur $\frac{m(m-1)}{2} = n = 3$; $A' = A = 0$; $A'' = AA' - 2B = 4$; $A''' = AA'' - BA' + 3C = 15$; $A^{IV} = 8$; $A^{V} = 50$, $A^{VI} = 91$. Hinc $a' = a = mA' = 12$; $a'' = mA^{IV} + \frac{6(A'')^2}{2} = 72$; $a''' = mA^{VI} + 15A''A^{IV} - \frac{20(A''')^2}{2} = -1497$; ac proinde $a = 12$; $b = \frac{aa' - aa''}{2} = 36$; $c = \frac{ba' - aa'' + a'''}{3} = -643$. Æquatio igitur (*D*) evadit
$v^3 - 12v^2 + 36v + 643 = 0$.
At illa æquatio non exhibet signa per vices positiva, & negativa; ergo proposita duas habet radices imaginarias (801.), adeoque unam tantum realem; ex quo conjicitur, numeros pro incognita x sufficiendos fore 0, 1, 2, 3, &c. (792. n. (1.)). Sumpta igitur x positiva, quum maximus coëfficiens negativus 2 unitate auctus, nempe 3, superet omnem radicem positivam (463.), hoc pro æquationis limite accepto, sat erit ponere successive $x = 0, 1, 2, 3$, factisque in hac ultima æquatione substitutionibus, numeri prodeuntes erunt -5, -6, -1, 16; ex

quo dignoscitur, realem propositæ radicem consistere inter 2, & 3, propterea que alterutrum ex hisce numeris fore valorem integrum huic radici proximum (803. n. 2.); Interea minorem limitem integrum 2 accipio (quod imposterum semper faciam), ac pono $x = 2 + \frac{1}{y}$ (814.), quare substituendo, ordinando per y, & mutando signa, ut primus terminus evadat positivus, provenit transformata

$y^3 - 10y^2 - 6y - 1 = 0.$

Hæc æquatio radicem habet unitate majorem (*l. cit.*) quamobrem numeri pro y substituendi, donec dicta signorum occurrat variatio, erunt 0, 1, 2, 3, &c. sed quoniam maximus coëfficiens negativus unitate auctus est 11, ex hoc substitutionem ordine retrogrado incipio, statuendo successive $x = 11$, 10, &c. atque habeo $+54$, -61, &c. ex quo concludo, valorem integrum proximum incognitæ y esse 10.

Ergo ob $q = 10$, statuo $y = 10 + \frac{1}{z}$, & obtineo æquationem

$61z^3 - 94z^2 - 20z - 1 = 0.$

Substituo ut supra successive $z = 1$, 2, &c. & statim habeo -54, $+71$, &c. hinc quoniam radix latet inter 1, & 2, erit $r = 1$; quare sumendo $z = 1 + \frac{1}{u}$, nanciscor $54u^3 + 25u^2 - 89u - 61 = 0$, ubi ponendo successive $u = 1$, 2, &c. provenient -71, $+293$, &c. adeoque $s = 1$.

Ita-

Itaque ſic proſequendo, reperientur numeri 2, 10, 1, 1, 2, 1, 3, 1, 1, 12, &c. proindeque radix quæſita exprimetur per fractionem continuam, nimirum

$$x = 2 + \cfrac{1}{10 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \&c.}}}}$$

unde devenietur ad fractiones communes

$$\frac{2}{1}, \frac{21}{10}, \frac{23}{11}, \frac{44}{21}, \frac{111}{53}, \frac{155}{74}, \frac{576}{275}, \frac{731}{349}, \frac{1307}{624}, \frac{16415}{7837},$$

&c. quæ per vices ſunt minores & majores quam valor incognitæ x, & idcirco ultima fractio eſt major radice quæſita, exiſtente tamen errore minore

quam $\frac{1}{(7837)^2}$ (772.), ſeu minore quam 0,0000000163.

Reducta igitur fractione $\frac{16415}{7837}$ ad fractionem decimalem, habetur 2,0945514865..... quamobrem radix quæſita ſtabit inter numeros 2, 09455149, & 2, 09455147.

Inventa radice reali, reliquæ duæ radices imaginariæ ſuperſunt indagandæ. Itaque in ſuperiori jam reperta æquatione $v^3 - 12v^2 + 36v + 643 = 0$, mutando v in $-w$, prodit

$w^3 + 12w^2 + 36w - 643 = 0$,

cujus æquationis quæri debet radix realis poſitiva (810.); at iſta ob ultimum terminum negativum

ibi necessario adest (432.); ergo ut ipsius valor proximus habeatur, substituendi successive sunt numeri naturales 0, 1, 2, 3, &c. (792. n. 1.) Sic si fiat $w = 6$, exurget -211, si ponatur $w = 7$, proveniet $+40$; quare valor integer proximus radicis quæsitæ erit 6. Statuendo igitur $w = 6 + \frac{1}{u}$, substituendo & mutando signa, obtinebitur æquatio

$$211u^3 - 216u^2 - 30u - 1 = 0,$$

quæ radicem realem unitate majorem comprehendet; ac proinde subrogando de more $u = 0, 1, 2,$ &c. prodibunt $-1, -36, +53$, adeoque 1 erit valor integer proximus radicis u. Ergo si ponatur $u = 1 + \frac{1}{x}$; factis substitutionibus, signisque mutatis, nova prodit æquatio

$$36x^3 - 171x^2 - 417x - 211 = 0.$$

Et rursus ponendo $x = 0, 1, 2,$ &c. signa negativa semper occurrent usque ad substitutionem $x = 7$, ex qua oritur $+9218$. Numerus igitur 6 erit valor integer proximus radicis x. Itaque si fiat $x = 6 + \frac{1}{y}$, & operationes ut supra continuentur, semper accedere pro libito possumus ad verum radicis w valorem, qui exprimetur per hanc fractionem continuam

$$w = 6 + \cfrac{1}{1 + \cfrac{1}{6 + \text{\&c.}}}$$

un-

unde communes colliguntur fractiones $\frac{6}{1}$, $\frac{7}{1}$, $\frac{48}{7}$, &c.

Cognita w, innotescet $\beta = \frac{\sqrt{w}}{2}$ (810.), & in æquatione proposita pro x subrogando $\alpha + \beta\sqrt{-1}$, nec non duas æquationes separatas formando, quarum altera terminos omnes reales, altera quantitate $\sqrt{-1}$ affectos habeat, duæ provenient æquationes

$\alpha^3 - (3\beta^2 + 2)\alpha - 5 = 0$

$3\alpha^2 - \beta^2 - 2 = 0$

quarum maximo communi divisore invento, continuetur divisio, donec ad residuum deveniatur, ubi x ad primum gradum evecta remaneat; hoc autem residuum erit $-\frac{8\beta^2 + 4}{3}\alpha - 5$, quo posito $= 0$, prodit $\alpha = \frac{-15}{4(2\beta^2 + 1)}$; atque ita duarum æquationis datæ radicum imaginariarum valor obtinebitur.

Exemplum II. Esto $x^3 - 7x + 7 = 0$.
Erit primo $m = 3$, ac proinde $n = 3$; præterea $A = 0$; $B = -7$; $C = -7$. Hinc $A = 0$; $A' = 14$; $A''' = -21$; $A^{\text{IV}} = 98$; $A^{\text{V}} = -245$; $A^{\text{VII}} = 833$; adeoque $a' = 42$, $a'' = 882$, $a''' = 18669$; ac tandem $a = 42$; $b = 441$, $c = 49$; quare æquatio (D) fit

$v^3 - 42v^2 + 441v - 49 = 0$.

Quum in hac æquatione signa sint per vices positiva, & negativa, ejus radices erunt omnes reales

 (796.);

(796.); quoniam vero iſta æquatio non eſt diviſibilis per v, nullas habebit radices invicem æquales (802.). Fiat igitur $v = \frac{1}{y}$ (814.), & nova æquatione ordinata per y, orietur

$$y^3 - 9y^2 + \frac{42}{49}y - \frac{1}{49} = 0.$$

In hac ob maximum cœfficientem negativum 9, ſumi pro limite poteſt 10 (463.), adeoque erit $\frac{1}{\sqrt{l}} = \frac{1}{\sqrt{10}} = \frac{1}{4}$ proxime; quare $K = 4$, & idcirco ex dictis (803. n. 2.) erit $\Delta = \frac{1}{4}$. In propoſita igitur $x^3 - 7x + 7 = 0$ ponatur $\frac{x}{4}$ pro x (804.), ac fiet

$$x^3 - 112x + 448 = 0.$$

Subſtitue de more 0, 1, 2, &c. pro x, & in iis, quæ prodibunt, duas tantum reperies ſignorum variationes, quæ reſpondebunt poſitionibus $x = 5, 6, 7$; adeo ut duæ tantum dentur radices poſitivæ, quarum altera latebit intra $\frac{5}{4}$, & $\frac{6}{4}$ altera intra $\frac{6}{4}$, $\frac{7}{4}$; ſic utriuſque radicis valor integer proximus erit 1, quia iſtæ duæ radices ſunt inter 1, & 2. Ut

Ut reliqua radix negativa indagetur, ponatur $-x$ pro x in proposita (790.), quæ proinde evadet $x^3 - 7x - 7 = 0$, quæque ob ultimum terminum negativum unicam comprehendet radicem positivam, quia reliquæ jam inventæ fuerunt; quamobrem substituendo successive pro x numeros 0, 1, 2, &c. signorum variatio inter substitutiones $x = 3$, & $x = 4$ reperietur, adeo ut statui possit 3 valor integer proximus radicis x in præcedenti æquatione, ac proinde -3 valor integer proximus radicis $-x$ in proposita.

Propositæ limitibus repertis, accedere pro arbitrio possumus ad verum radicum valorem. Et quoad radices positivas, ponamus in æquatione $x^3 - 7x + 7 = 0$, ut moris est, $x = 1 + \frac{1}{y}$ ipsa convertetur in sequentem

$y^3 - 4y^2 + 3y + 1 = 0$,

sed quia 1 est valor proximus integer duarum radicum, duas æquatio ista habebit radices unitate majores (817.). Ut ad harum radicum verum valorem propius accedam, ob 4 maximum coëfficientem negativum, pro y successive substituo 0, 1, 2, 3, 4, 5, ex quo colligo numeros 1, 1, -1, 1, 13, 41; hinc coniicio, alteram radicum quæsitarum latere inter 1, & 2, alteram inter 2, & 3, adeoque valores proximos incognitæ y esse 1, & 2. Itaque

(1.°) Statuo $y = 1 + \frac{1}{z}$, unde habeo

$$z^3 - 2z^2 - z + 1 = 0,$$

quæ

quæ æquatio unam tantum continet radicem realem unitate majorem. Fiat igitur successive $z = 1, 2$, &c. usque ad signorum in substitutionum operationibus mutationem, quod evenit cum ponuntur $z = 2$, & $z = 3$, in primo enim casu habetur $- 1$, in altero $+ 7$; Ergo 2 erit valor integer proximus incognitæ z. Fac $z = 2 + \frac{1}{u}$, & substituendo, nec non signa mutando, prodibit

$u^3 + 3u^2 - 4u - 1 = 0$.

Rursus statue $u = 1, 2$, &c. & invenies, valorem integrum proximum incognitæ u esse 1; quare pone $u = 1 + \frac{1}{w}$, & ita porro

(2.°) Esto $y = 2 + \frac{1}{z}$, & factis in superiori æquatione $y^3 - 4y^2 + 3y + 1 = 0$ substitutionibus, post signorum mutationem proveniet

$z^3 + z^2 - 2z - 1 = 0$.

Ista etiam æquatio unicam radicem realem unitate majorem complectetur, quapropter fiat de more $z = 1, 2$, &c. & ex substitutionibus $z = 1$, $z = 2$ emerget $- 1$, $+ 5$; ex quo infertur, esse 1 valorem integrum proximum incognitæ z. Fiat igitur $z = 1 + \frac{1}{u}$, & post signa mutata exurget

$u^3 - 3u^2 - 4u - 1 = 0$.

Hic (agendo ut supra) invenies, valorem integrum proximum incognitæ u esse 4; nam $u = 4$ dat

dat -1, & $u=5$ præbet $+29$. Itaque ſtatue $u=4+\frac{1}{w}$; & ſic deinceps.

Ergo duæ radices quæſitæ exprimentur per

$$x=1+\frac{1}{1+\frac{1}{2+\frac{1}{1+\&c.}}}$$

$$x=1+\frac{1}{2+\frac{1}{1+\frac{1}{4+\&c.}}}$$

unde obtineri poſſunt ſeries convergentes, ut in Exemplo antecedente.

Ad valorem proximum radicis negativæ inveſtigandum revoco æquationem

$$x^3-7x-7=0,$$

cujus valor proximus integer eſt 3; quare fiat $x=3+\frac{1}{y}$, quod ſigna mutando exhibet

$$y^3-20y^2-9y-1=0.$$

Ob maximum cœfficientem negativum 20 experior, an ponendo $x=21, 20, 19$, &c. offendere poſſim poſt ſubſtitutionem operationem ad ſignorum diverſitatem, & reperio, valorem proximum quæſitum eſſe 20. Itaque ſtatuo $y=20+\frac{1}{u}$, &c. ſic ra-

radix negativa æquationis propositæ designabitur per

$$x = -3 - \cfrac{1}{20 + \cfrac{1}{3 + \&c.}}$$

822. *Scholion.* Hic oblata occasione, sequens Problema, velut in sua sede locatum, quamvis ad Caput antecedens pertineat, subjungendum censeo; nimirum

Data æquatione qualibet, cujus radices reales sint cum imaginariis commixtæ, imaginariarum numerum invenire.

Resolutio. Si radices reales sint cognitæ, res patebit; at si ignorentur, quoniam radices invicem æquales, si quæ adsint in æquatione data, per æquationem (D) statim dignoscuntur (802.), hisce per traditas methodos repertis (514.), & per divisionem a proposita sublatis, æquationis superstitis quærantur limites (803.), qui, debitis cautionibus adhibitis, debent omnes in conspectum venire. Sed per hosce limites, vel ipsæ radices reales, si fuerint rationales, vel earum numerus per totidem signorum variationes, si fuerint irrationales, necessario innotescent. Ergo reliquæ radices, quarum numerus numerum radicum in proposita existentium complere debet (407.), erunt imaginariæ. *Quod erat* &c. Prolixus quidem in altioribus æquationibus evadit calculus, quod æquatio dimensionis μ operationes 2μ requirit ad eliciendos valores omnes quantitatum

A'', A''', &c. usque ad $A^{2\mu}$, qui quantitatibus a', a'', &c.

&c. usque ad a^{u} respondent (798.), ut æquatio $v^n - av^{n-1} + bv^{n-2} - cv^{n-3} + \&c. = 0$ cum suorum coëfficientium valoribus habeatur; quod tamen generali huic methodo, quamvis indirectæ, minime adversatur.

CAPUT IX.

DE SUMMA FUNCTIONUM UNICAM VARIABILEM COMPLECTENTIUM PER SERIES FINITAS AUT INFINITAS EXHIBENDA.

Definitiones.

823. Curvæ alicujus *MAM diameter* AB (*Fig.* 25.), bifariam secat parallelas illas *MM*, *MM*, &c. quæ perimetro ejusdem curvæ sint utrinque terminatæ. Dicitur vero *axis* cum ipsas secat ad angulos rectos.

824. Curvæ *vertex* dicitur punctum illud *A* in perimetro curvilineo, ex quo diameter vel axis *AB* incipit.

825. *Ordinatim applicatæ* vocantur lineæ æquidistantes *MM*, *MM*, &c. bifariam sectæ a diametro vel axi *AB*, earumque partes dimidiæ *PM*, *PM*, &c. *semiordinatæ* appellantur, quamvis etiam *ordinatæ*, vel *applicatæ* vocitentur, considératæ ac si forent integræ.

826. *Abscissa* ea nuncupatur diametri vel axis portio indeterminata *AP*, inter perimetrum curvilineum & ordinatim applicatam intercepta.

827.

827. *Coordinatæ* appellantur abſciſſa, & ordinatim applicata ipſi reſpondens ſimul conſideratæ.

828. *Curva algebrica* illa dicitur, cujus æquatio per coordinatas accurate exprimi poteſt.

829. Propoſitio I. *Lineæ curvæ ſunt veluti polygonorum perimetri conſiderandæ.*

Dem. Cum recta curvam tangit, ajunt, contactum fieri in puncto. Quæro igitur, utrum hoc punctum ſit nihilum abſolutum, an aliquid? Si primum, tangens non tanget, quod eſt abſurdum. Si ſecundum, contactus quantitas vel eſt infinite parva abſoluta, vel relativa; at infinite parvum abſolutum non datur, ut ſeries in infinitum decreſcens demonſtrat; ergo contactus quantitas eſt portio immane parva, ſed realiter exiſtens. Fac modo, ut alius tangentis contactus ſit primo contactu conterminus. Duæ quantitates a duobus hiſce contactibus efformatæ non erunt in directum, ſecus ſic proſequendo linea tacta non foret curva ſed recta. Ergo ſi talis operatio continuetur, latera alicujus polygoni neceſſario prodibunt, adeoque linea curva eſſe nequit niſi laterculorum angulos obtuſiſſimos efficientium congeries, quod oſtendit, omnes curvas eſſe veluti totidem polygonorum perimetros conſiderandas. *Quod erat* &c.

830. Propoſitio II. *Curva qualibet conflatur ex rectangulis latitudinis evaneſcentis.*

Dem. Eſto curva *mM* (*Fig.* 26. 27.), cujus axis *AP*. Ducantur duæ ordinatæ *PM*, *pm* laterculum *Mm* intercipientes, & ex puncto *M* demittatur ſupra *pm* normalis *Mo*, qua biſariam diviſa in *e*, per *e*

aga-

agatur alia ordinata *qu* alterutri ipsarum *PM*, *pm* parallela. Triangulum *Mom* erit ad rectangulum *pM*, ut *en* ad *PM* Rursus secta bifariam *Me*, concipiatur nova ordinata ducta per punctum hujus divisionis. Triangulum *Men* erit ad rectangulum *qM*, ut $\frac{1}{2}$ *en* ad *PM*. Continuando successive hanc divisionem in recta *Mo*, devenies tandem ad portionem ipsius *om* quacumque data minorem; quae erit altitudo trianguli, cui respondebit rectangulum latitudinis evanescentis. Quamobrem quum triangulum istud sit ad datum rectangulum evanescens, ut ultimae portionis inassignabilis dimidium ad *PM*, triangulum illud pro nihilo haberi poterit. Ergo si totum spatium curvilineum in hujusmodi rectangula latitudinis evanescentis resolvi intelligatur, ex iis omnino coalescens statui tutissime poterit. *Quod erat* &c.

831. *Monitum.* Si sumpto numero quolibet positivo *m* integro vel fracto, positaque recta quavis constante $= 1$, sit cujuscumque ordinatae *PM* (*Fig.* 26.) vel $pm = y$ potestas *m* aequalis vel proportionalis respondenti abscissae *AP*, vel *Ap* ductae in dictam constantem $= 1$, adeo ut ubique habeatur aequatio $y^m = x$; curvae *AM*, quae hinc orientur, appellantur *Parabolae*, quarum species pro numeri *m* diversitate variant in infinitum; & quidem si fuerit $m = 2$ (posito enim numero $m = 1$, fit triangulum), curva *AM* sub aequatione $y^2 = x$ dicitur pa-

parabola primi ordinis, seu *quadratica*, seu *apolloniana*; si $m = 3$, ita ut exurgat æquatio $y^3 = x$, curva AM vocatur *parabola cubica*, seu *secundi ordinis*, & sic de ceteris. Si m fuerit numerus fractus, parabola hinc prodiens dicitur genus aliquod *paraboloidum*, vel curvarum parabolis *agnatarum*; quantitas vero constans $= 1$, *parameter* nuncupatur. Quod si m fuerit numerus integer vel fractus negativus, ita ut sit semper potestas y^m reciproca respondenti abscissæ x, velut $PM^m : pm^m = Ap : AP$ (*Fig.* 27.), curva AM erit aliqua ex genere *Hyperbolarum intra asymptotos*, seu rectas AT, AV, quæ curvas nunquam tangunt, quamvis in infinitum protensæ. Sic si $m = 1$, curva Mm erit *hyperbola primi ordinis*, seu *apolloniana intra asymptotos*, & ita porro.

832. Propositio III. *Infinitas parabolas, & infinitas hyperbolas intra asymptotos quadrare.*

Resolutio. Exprimat curva Mm (*Fig.* 26.) quascumque parabolas, & (*Fig.* 27.) quascumque hyperbolas intra asymptotos; sitque ut supra $AP = x$, $PM = y$, & æquatio $y^m = x$, seu $y = x^{\frac{1}{m}} = \sqrt[m]{x}$. Resolvatur tota area APM in trapetia, seu potius rectangula Pm latitudinis evanescentis (830.), & ejusdem altitudinis Pp, quæ fiat $= e$. Primum rectangulum ex recensitis, incipiendo a vertice A, erit $e\sqrt[m]{e}$, tunc enim $y \times e = \sqrt[m]{e} \times e$; secundum re-

rectangulum erit $y \times e = \sqrt[m]{2e} \times e$, tertium $y \times e = \sqrt[m]{3e} \times e$, & sic prosequendo; quare (721.) spatium indeterminatum $APM = e\sqrt[m]{e} + e\sqrt[m]{2e} + e\sqrt[m]{3e} + \ldots + e\sqrt[m]{x} = e\sqrt[m]{e}\left(1^{\frac{1}{m}} + 2^{\frac{1}{m}} + 3^{\frac{1}{m}} + \ldots + \left(\frac{x}{e}\right)^{\frac{1}{m}}\right) = \frac{m}{m+1} e^{\frac{m+1}{m}}\left(x^{\frac{m+1}{m}}\right)\left(e^{\frac{-(m+1)}{m}}\right) =$

$\frac{m}{m+1} x^{\frac{m+1}{m}}$. Sed $x^{\frac{m+1}{m}} = x \times x^{\frac{1}{m}} = xy$. Ergo $\frac{m}{m+1} xy$

$= APM$, si agatur de parabolis; & $\frac{-m}{1-m} xy$, vel $\frac{m}{m-1} xy = APMV$, si de hyperbolis intra asymptotos. *Quod erat* &c.

833. *Coroll.* Ergo si $m = 2$, erit $y^2 = x$, æquatio nempe ad parabolam primi ordinis, cujus quadratura $= \frac{2}{3} xy$. Si $m = \frac{2}{5}$, erit $y^{\frac{2}{5}} = x$; rursus ad paraboloidem, cujus quadratura $= \frac{2}{7} xy$; &c.

Quoad hyperbolas intra asymptotos si $m > 1$, spatii indeterminati $AVMP$ (*Fig.* 27.) summa, seu quadratura erit in promptu. Si $m < 1$, summa, quæ semper quoque habebitur, non ad spatium $AVMP$,

 sed

sed ad oppositum PTM, ob signum negativum exurgens, refertur. Si $m = 1$, neutrum spatium $\left(\text{ob } \frac{1}{1-1} = \frac{1}{0}\right)$ quadratur. Sic si fuerit $y = x^{-\frac{1}{2}}$, erit $m = 2$, adeoque (statuendo $\int$ pro summæ signo), erit $\int \frac{1}{\sqrt{x}} = 2xy = 2\sqrt{x}$. Si $y = x^{-\frac{3}{4}}$, erit $m = \frac{4}{3}$, ex quo fit $\int x^{-\frac{3}{4}} = \frac{m}{m-1} xy = 4xy = 4\sqrt[4]{x}$. Si $y = x^{-2}$, ob $-\frac{1}{m} = -2$, & $m = \frac{1}{2}$, erit $\int x^{-2} = -xy = -\frac{1}{x}$; similiter si $y = x^{-3}$, ob $m = \frac{1}{3}$, erit $\int x^{-3} = -\frac{1}{2} xy = -\frac{1}{2x^2}$; vel si ponatur $n = \frac{1}{m}$, erit $\int x^{-n} = -\frac{1}{(n-1)x^{n-1}}$.

834. Propositio IV. *Spatii curvilinei dimensio haberi quoque potest, si ordinatim applicatarum idem spatium penitus implentium summa per seriem finitam aut infinitam obtineatur.*

Dem. Spatio curvilineo APM (*Fig.* 26.) circumscribe parallelogrammum rectangulum PQ, tum concipe, describi sub æqualibus altitudinibus omnia dicta rectangula evanescentia areas utriusque figuræ prorsus complentia. Erunt invicem hæ figuræ, ut re-

respectivæ istorum rectangulorum summæ. Sed omnia rectangula spatii curvilinei sunt ad omnia rectangula parallelogrammi circumscripti, ut omnes ordinatim applicatæ primæ figuræ ad omnes ordinatim applicatas secundæ, omnes enim sunt per eandem altitudinem multiplicatæ; ergo dictum spatium curvilineum est ad parallelogrammum circumscriptum, ut omnes ordinatim applicatæ primi spatii ad omnes secundi. At si in parallelogrammo omnium ordinatim applicatarum (quarum numerus per altitudinem exprimitur) summa obtineatur, ejus dimensio innotescit; ergo &c. *Quod erat* &c.

835. *Coroll.* 1. Hinc si omnes *PM* spatium quodvis rectilineum, vel curvilineum replentes ordinatim applicentur cuicumque lineæ curvæ sub eadem vel æquali altitudine, hujusmodi spatia quamvis aspectu inter se dissimilia, erunt æqualia, utraque namque iisdem singillatim elementis eodem ordine dispositis concinnatur.

836. *Coroll.* 2. Parabolas & hyperbolas intra asymptotos possumus ex præsenti ratiocinio brevius quadrare. Etenim concipe parabolæ, seu hyperbolæ cujuscumque ordinis axem divisum esse in portiones evanescentes invicem æquales, positisque, ut paulo ante singulis istis portionibus $= e$, singulisque ordinatim applicatis respondentibus $= y = n^{\frac{1}{r}} e^{\frac{1}{r}}$, habemus primam applicatam $= e^{\frac{1}{r}}$, secundam $2^{\frac{1}{r}} e^{\frac{1}{r}}$, tertiam $3^{\frac{1}{r}} e^{\frac{1}{r}}$, &c. unde seriei $e^{\frac{1}{r}} \left(1^{\frac{1}{r}} + 2^{\frac{1}{r}}\right.$

$2^{\frac{1}{r}} + 3^{\frac{1}{r}} +$ &c. in infinitum) summa erit factum ex $\frac{r}{1+r}$ in numerum omnium $e^{\frac{1+r}{r}}$ (721.). Sed omnium e numerus est immane magnus ob dictam axis divisionem in portiones evanescentes; Ergo hæc summa evadet $\frac{r}{1+r} n^{\frac{1+r}{r}}$, quæ cum antecedente conspirat.

837. Propositio V. *Si fuerint quæcumque figuræ sequentes, videlicet* rectangulum, triangulum rectangulum isosceles, & aliqua tam ex parabolis, quam ex hyperbolis intra asymptotos cujuscumque ordinis sub æquali altitudine, *æquationem ad basce figuras in unico spatio simul comprehensas invenire.*

Resolutio. Ex indefinitis hujusmodi figurarum axibus concipiantur sectæ ad arbitrium portiones æqualis altitudinis, quibus singulis propriæ ordinatim applicatæ respondeant. Quoniam in rectangulo tales applicatæ sunt constantes, earum quælibet fiat $= a$; quum autem in triangulo rectangulo isoscele quævis applicata sit abscissæ æqualis, ponatur hæc $= x$; applicata t alicujus parabolæ expressa per axem erit $x^{\frac{1}{m}}$, pariterque applicata v alicujus hyperbolæ intra asymptotos indicabitur per $x^{-\frac{1}{m}}$. Itaque si rectæ a, x, t, v sumptæ sub dicta altitudine, quæ velut abscissa communis consideratur, intelligantur positæ in directum, ita ut formetur recta, quæ sit una ex or-

ordinatim applicatis ejusdem abscissæ, voceturque y, erit $y = a + x + x^{\frac{1}{m}} + x^{-\frac{1}{m}}$. Consimilis operatio instituatur, si plures sint hujusmodi figuræ, nimirum si fuerint duo rectangula inæqualia, tria triangula rectangula isoscelia, una ex parabolis quadraticis concavis versus axem, duæ ex cubicis versus axem convexis, una ex hyperbolis secundi generis intra asymptotos, habebitur $y = a + b + 3x + x^{\frac{1}{2}} + 2x^3 + x^{-\frac{1}{2}}$. Quod si ex æquatione $y = a + x + x^2 + x^{-\frac{1}{3}}$ aliquam id genus figuram, puta parabolam cubicam versus axem concavam subtrahere cupias, æquatio evadet $y = a + x + x^2 + x^{-\frac{1}{3}} - x^{\frac{1}{3}}$, & sic de aliis.

Jam vero si finges, sumptas esse alias altitudines invicem æquales axium memoratorum, & respondentes applicatæ in una recta rite distributæ sub eadem altitudine in dicta abscissa ordinentur, & hoc in quolibet altitudinis accepto puncto, linea per omnes talium rectarum inæqualium extremitates transiens figuram curvilineam, vel mixtilineam efficiet, quæ eandem semper retinebit æquationem. *Quod erat* &c.

838. Propositio VI. *Æquationum algebricarum, in quarum altero membro stet* y, *in altero functio integra, vel fracta ipsius* x, *summam per seriem finitam aut infinitam indagare.*

Resolutio. Quoniam res eo deducitur, ut singulorum terminorum dictam functionem comprehendentium, & figuras geometricas indicantium summa reperiatur:

I. Si functio axis x sit rationalis, ita ut unusquisque terminus ejusdem sit monomium, imaginare (coëfficientibus prætermissis) tot æquationes ad variabiles, quot sunt termini in dicta functione. Quum autem æquationes istæ reducantur ad triangulum rectangulum isosceles, si adsit variabilis, cujus exponens sit unitas; ad aliquam ex infinitis parabolis si adsit ipsa variabilis ad aliquam potestatem elevata; ad aliquam ex infinitis hyperbolis intra asymptotos, si adsit unitas divisa per eandem variabilem ad potestatem aliquam evectam; accipe summas singulorum hujusmodi terminorum, videlicet In casu quantitatis constantis (quam omisimus, quia non habet æquationem), quum ipsa exprimat rectangulum cum abscissa, summa erit ax.

In casu solius variabilis x, quum hæc exprimat triangulum rectangulum isosceles, cujus unum crus est abscissa, ejus summa erit $\frac{x^2}{2}$.

In casu parabolarum & hyperbolarum intra asymptotos, earum summam vidimus esse $\frac{m}{m+1} x^{\frac{m+1}{m}}$, ubi m est quantitas affirmativa pro parabolis, negativa pro hyperbolis.

In casu vero hyperbolæ primi ordinis ubi $m = 1$, sum-

umma quoti $\frac{1}{x}$ est lx, ut infra ostendetur.

II. Si dicta functio habeat radicales variabilis monomiæ, veluti $\sqrt{x}, \sqrt[m]{x}, \frac{1}{\sqrt{x}}, \frac{1}{\sqrt[m]{x}}$, summa singulorum terminorum reducetur ad parabolarum, & hyperbolarum intra asymptotos quadraturam, ut supra.

III. Si dicta functio sit quantitas rationalis polynomia fracta, sive radicalis polynomia integra, vel fracta, resolvatur in seriem prout jam docuimus in Cap. VI. Sect. I. & in Cap. II. Sect. IV. tum hujus seriei termini in totidem summas, instar num. I. & II. reducantur.

IV. Si variabilis sit per coëfficientes multiplicata aut divisa, nullam isti patientur mutationem, ut apertum est. *Quod erat* &c.

839. *Coroll.* 1. Quum functio quævis ipsius variabilis summam finitam non admittens sit resolubilis in seriem recurrentem, patet, infinitarum serierum recurrentium, quamvis unaquæque sit ad infinitum vergens, summas per series infinitas haberi posse.

840. *Coroll.* 2. Quicumque sit exponens, ad quem sit elata variabilis x, ejus summa est generatim $\frac{m}{m+1} x^{\frac{m+1}{m}}$, ubi pro m intelligitur numerus positivus, aut negativus, integer, vel fractus; nimirum *variabilis exponens augeatur unitate, tum per eundem sic auctum exponentem ipsa variabilis divi-*

datur; & hoc est fundamentum *Calculi Integralis*, seu *Summatorii*.

841. *Coroll.* 3. Vicissim ut habeatur functio, ex qua summa proficiscitur, *multiplicetur variabilis per suum exponentem, tum exponens variabili eidem imminens unitate imminuatur*. Sic si fuerit summa $\frac{m}{m+1} x^{\frac{m+1}{m}}$, divide per $\frac{m}{m+1}$, deinde ab exponente $\frac{m+1}{m}$ dene unitatem, & habebis $x^{\frac{1}{m}}$; similiter si generatim fuerit summa x^n, ubi n sit numerus integer, vel fractus, positivus, aut negativus, functio, unde hæc summa oritur erit nx^{n-1}. Et hoc est fundamentum *Calculi Differentialis*.

842. *Scholion*. Hæc methodus transformationem variabilis adsumptæ, puta x, in aliam variablem quantitate aliqua auctam vel imminutam non admittit, per hanc enim transformationem curva in aliam priori diversam transit, in qua elementa diverso modo progrediuntur, quare conditiones recensitæ tunc locum amplius non obtinent. Sic si curva *AM* (*Fig.* 26.), in qua sit $AP = x$, $PM = y$, definiatur per æquationem $y = \frac{x}{b}\sqrt{x^2+a^2}$, fiatque $\sqrt{x^2+a^2} = v$, erit $v^2 = x^2 + a^2$, & $x^2 = v^2 - a^2$, quæ æquatio pertinet ad hyperbolam æquilateram

ram BN (*Fig.* 28.), cujus axis transversus $DB = 2a$, abscissæ CQ a centro C incipientes $= v$, semiordinatæ $QN = u$, ut norunt qui sectiones Conicas vel leviter degustarunt. Posita enim $BQ = z = QC - CB = v - a$, quum ex dictæ hyperbolæ natura sit $u^2 = 2az + z^2$, substituendo $v - a$ pro z, & $v^2 - 2av + a^2$ pro z^2, prodit $u^2 = v^2 - a^2$. Quare abscissa $AP = u$ curvæ AM (*Fig.* 26.), cujus quæritur quadratura, applicatur ad hyperbolam BN (*Fig.* 28.) per semiordinatam QN, cui respondet $CQ = v$. Hæc igitur curva est toto cælo diversa a curva AM, simulque ab istis diversæ sunt illius abscissæ, semiordinatæ, & altitudines.

Similiter si proponatur æquatio $y = u\sqrt{u+a}$, fiatque $\sqrt{u+a} = v$; erit $u + a = v^2$, & $u = v^2 - a$. Quamobrem curva transformata erit $y = v^3 - av$. At hæc nova curva diversa est a proposita, adeoque quum requisitæ conditiones hic quoque deficiant, methodus allata non est adhibenda, quod semper advertatur.

Exemplum I. Esto $\frac{1}{1+u}$, sitque $u > 1$. Hæc functio resolvatur in seriem: proveniet $1 - x + u^2 - x^3 + u^4 + \&c.$ cujus summa $= u - \frac{1}{2}u^2 + \frac{1}{3}u^3 - \frac{1}{4}x^4 + \&c.$ sed hæc summa, ut alibi vidimus (257.) adæquat $l(1+x)$; ergo $\int \frac{1}{1+u} = l(1+u)$.

843. *Coroll.* 1. Si in denominatore $1+x$ imminues successive unitatem, donec ista evadat evanescens, hæc unitas pro nihilo haberi poterit; sed tunc quantitas $1+x$ fit $=x$; ergo $\int\frac{1}{x}=lx$.

844. *Coroll.* 2. Sit (*Fig.* 29.) quadratum CE, cujus latus $CA=1$, potentia hyperbolæ æquilateræ EM, sitque $AP=x$, $PM=y$, æquatio ad portionem hyperbolicam $APME$ erit $y=\frac{1}{1+x}$, sumptisque omnibus ordinatim applicatis y, earum, adeoque portionis indeterminatæ $APME$ summa erit $\int\frac{1}{1+x}=l\ (1+x)$.

845. *Coroll.* 3. Vicissim constat ex Conicis, spatium hyperbolicum $APME$ exprimere logarithmum numeri $CA+AP$, seu esse $\int\frac{1}{1+x}=l(1+x)$; ergo $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\&c.=l(1+x)$.

Ex quibus apparet, series divergentes non esse omnino spernendas.

Exemplum II. Si detur æquatio $y=\frac{1+2x}{1-x-xx}$; functio $\frac{1+2x}{1-x-xx}$ in seriem conversa evadet $1+3x$

+

$+4x^2+7x^3+11x^4+18x^5+$&c. quare $\int\left(\frac{1+2x}{1-x-xx}\right)$

$=x+\frac{3}{2}x^2+\frac{4}{3}x^3+\frac{7}{4}x^4+\frac{11}{5}x^5+\frac{18}{6}x^6+$ &c.

Similiter si fuerit $y=\frac{1}{\sqrt{1+xx}}$, hac functione in seriem resoluta, prodit $1-\frac{1}{2}x^2+\frac{1\cdot3}{2\cdot4}x^4-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}x^6+$ &c. adeoque $\int\frac{1}{\sqrt{1+xx}}=x-\frac{1}{2\cdot3}x^3+\frac{1\cdot3}{2\cdot4\cdot5}x^5-\frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot7}x^7+$ &c.

Exemplum III. Si circuli diameter AB (*Fig.* 30.) ponatur $=1$, $AP=x$, $PM=y$, adeoque $PB=1-x$; notum est, ejus æquationem esse $y^2=x-xx$, sive $y=\sqrt{x-xx}$. Resolvatur $(x-xx)^{\frac{1}{2}}$ in seriem; fiet $(x-xx)^{\frac{1}{2}}=x^{\frac{1}{2}}-\frac{1}{2}x^{\frac{3}{2}}-\frac{1}{2\cdot4}x^{\frac{5}{2}}-\frac{1}{4\cdot7}x^{\frac{7}{2}}-\frac{1\cdot3}{4\cdot6\cdot9}x^{\frac{9}{2}}-\frac{1\cdot3\cdot5}{4\cdot6\cdot8\cdot11}x^{\frac{11}{2}}-$ &c. quamobrem $\int(x-xx)^{\frac{1}{2}}=\frac{2}{3}x^{\frac{3}{2}}-\frac{1}{5}x^{\frac{5}{2}}-$

$\frac{1}{4.7} x^{\frac{7}{2}} - \frac{1.3}{4.6.9} x^{\frac{9}{2}} - \frac{1.3.5}{4.6.8.11} x^{\frac{11}{2}} -$ &c. $=$

$\sqrt{x} \left(\frac{2}{3} x - \frac{1}{5} x^2 - \frac{1}{28} x^3 - \frac{1}{72} x^4 - \frac{1}{704} x^5 - \right.$

$\left. \frac{1}{1664} x^6 - \&c. \right)$ quæ ſumma ſegmentum indeterminatum APM repræſentat.

Simili modo ſi radius circuli CA ſtatuatur $= 1$; alia circuli æquatio, abſciſſis a centro computatis, erit $y^2 = 1 - xx$, ſive $y = \sqrt{1 - xx} = 1 - \frac{1}{2} x^2 - \frac{1}{8} x^4 - \frac{1}{16} x^6 - \frac{5}{128} x^8 - \frac{7}{256} x^{10} -$ &c. hinc

$\int \sqrt{1 - xx} = x - \frac{1}{6} x^3 - \frac{1}{40} x^5 - \frac{1}{112} x^7 - \frac{5}{1152} x^9 - \frac{7}{2816} x^{11} -$ &c. Cum igitur x evadit æqualis radio $= 1$, ſpatium indeterminatum $CDMP$ per hanc ſummam expreſſum tranſit in quadrantem ACD.

Eſto rurſus AE dimidium arcus AM, & ejus tangens $AT = x$, tangens At arcus dupli AM erit $= \frac{2x}{1 - xx}$ (228.), junctiſque TC, tC, fiet $AT : Tt =$ $AC : tC$, hoc eſt $x : \frac{x + x^3}{1 - xx} = 1 : \frac{x + xx}{1 - xx}$, quod Tt

$=$

$= \frac{2x}{1-xx} - x = \frac{x+x^3}{1-xx}$. Præterea $Ct : At =$ $CM : PM$, seu $\frac{1+xx}{1-xx} : \frac{2x}{1-xx} = 1 : \frac{2x}{1+xx}$.

Sed $2x(1+x)^{-2} = 2x(1-x^2+x^4-x^6+x^8-x^{10}+$ &c.); ergo hujus seriei summa $= x^2(\frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} +$ &c.) $=$ segm. $APM = x^3(1 - \frac{1}{3}x^2 + \frac{1}{5}x^4 - \frac{1}{7}x^6 +$ &c.). Itaque si AE sit arcus 45°, ejus duplum erit AD, quarta nimirum pars totius circumferentiæ. Tunc autem (dictis analogiis semper manentibus, quamvis tangens At evadat infinita), PM degenerat in radicum CD, & AT $= x$ fit $= CD = 1$; quare quadrans ACD evadit $= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} +$ &c.

846. *Scholion.* Possem per hanc methodum longius procedere, verum ista & alia id genus longe præstantiora in Operibus Calculum infinitesimalem adornantibus omnibus numeris absoluta reperientur. Mihi suffi:it in præsentiarum eo Algebram Cartesianam deduxisse, ut Calculi Differentialis, & Integralis doctrinam primis tantum labiis Tiro attinge-

gere possit, unde facilior ad sublimiora ipsi aditus pateat. Interea ut ad circulum redeam, observatione dignum mihi sequens videtur

Theorema. *Si arcus circularis* ABB *ratio ad integram circumferentiam* APND, *cujus centrum* L, *innotescat, & C sit centrum gravitatis ejusdem arcus; dico, circumferentiam, quæ priori* APND *concentrica per punctum* C *transit, esse geometrice rectificabilem.*

Etenim finge, semicirculum *MDN* rotari circa diametrum *MN*, superficies sphærica genita ab arcu *ADB* erit per Archimedem æqualis producto ex tota circumferentia *APND* in chordam *AB*. Sed hæc sphærica superficies æquatur per Regulam Guldinianam facto ex arcu *ADB* in viam sui centri gravitatis; Ergo ut arcus *ADB* ad integram circumferentiam, ita erit chorda ad dictam viam centri gravitatis. At ratio arcus *ADB* ad suam circumferentiam ponitur cognita; ergo via sui centri gravitatis, quæ circulum efficit, innotescet. Quod si dictus arcus in semicircumferentiam degeneret, circumferentia a via centri gravitatis *C* efformata erit dupla diametri *MN*.

Hinc habetur centrum gravitatis semicircumferentiæ, si fiat ut circumferentia ad diametrum, ita radius ad distantiam inter centrum circuli & centrum gravitatis ipsius circumferentiæ. Generatim ut arcus *ADB* ad chordam *AB*, ita radius *ML* ad *LC*. Haud absimili modo *si cognita sit ratio Sectoris circularis* LADB, *cuius centrum gravitatis* C, *ad integram circumferentiam* APND, *circumferentia illa, quæ*

ipsi

ipsi APND *concentrica punctum* C *traiicit*, *demonstrabitur geometrice rectificabilis.*

Solidum enim genitum a rotatione sectoris $LADB$ circa diametrum MN est æquale superficiei ejusdem ductæ in trientem radii AL, quod etiam ex eo constat, quia solidum istud resolvi potest in innumeras pyramides immane parvas, quarum vertices sint in centro L, bases in dicta superficie sphærica. At superficies ista sphærica est per Archimedem æqualis circumferentiæ $DMPN$ ductæ in chordam AB. Ergo solidum, de quo est quæstio, adæquat productum ex circumferentia integra $DMPN$ in AB per $\frac{1}{3}$ AL multiplicatum. At hoc idem solidum est æquale sectori $LADB$ ducto in viam sui centri gravitatis; adeoque ut sector $LADB$ ad productum ex circumferentia $DMPN$ in AB, ita $\frac{1}{3}$ AL ad circumferentiam, quam via centri gravitatis, dum peragitur dicta revolutio, describit.

Si vero sector in semicirculum migret, tunc analogia erit, ut semicirculus ad integram sphæræ superficiem ab ipso genitam, sive ut 1 ad 8, ita $\frac{1}{3}$ AL ad viam centri gravitatis ejusdem semicirculi, quæ erit $\frac{8}{3}$ radii AL, seu $\frac{4}{3}$ diametri MN; quod sic

quo-

quoque demonſtratur. Sphæra $DMPN$ oritur ex ſemicirculo MDN ducto in $\frac{4}{3}$ MN; ſed ſphæra eſt etiam æqualis facto ex eodem ſemicirculo in viam centri gravitatis C; ergo &c.

Hinc ſi fiat ut arcus AD ad $2AB$, ita $\frac{1}{3}$ $ML:LC$, erit C centrum gravitatis ſectoris $LADB$; quod ſi ſector in ſemicirculum degeneret, erit arcus $45°$ ad diametrum; ut triens radii ad diſtantiam inter centrum circuli & centrum gravitatis ſemicirculi.

AD·

ADDITAMENTUM I.

Quamquam præclarissima Ingenia in Calculi Differentialis principiis ostendendis excelluere, nondum tamen simplicissimam eorum demonstrationem, ni fallor, sunt adsecuti. Itaque operæ pretium existimo eadem in Tironum gratiam a limpido ac genuino suo fonte derivare; quum enim Algebra sit Logysticæ, seu Logarithmicæ superstructa, hac in curva, ejusque functionibus hujusmodi elementa sunt inquirenda.

Aliam postea methodum tradam a vulgata in isto Calculo tractando aliquantum diversam, cujus ope nonnulla problemata difficilioris indaginis resolvuntur. Sit igitur

Propositio I. Fundamentalis

Differentiale quantitatis ly, *vel* l(A ± y) *invenire.*

Resolutio. Notum est, etiam ex Geometricis, Logysticæ subtangentem esse constantem. (Vide inter alios cel. Keilium *de logarithmis* Cap. V.); æquatio vero ad Logysticam est $x = ly$, sive $x = l(A \pm y)$. Sed posita dicta subtangente $= 1$, per triangulum Barrowianum stat $dy : dx = y : 1$, vel $\pm dy : dx = A \pm y : 1$; ergo $dx = \frac{dy}{y}$, vel $dx = \pm \frac{dy}{A \pm y}$. Est autem dx differentiale tam ipsius x, quam alterius ei-

eidem æqualis ly, sive $l(A \pm y)$. Ergo differentiale logarithmi ejusdem y, hoc est $d.ly = \frac{dy}{y}$ in primo casu; & differentiale logarithmi quantitatis $A \pm y$, seu $d.(A \pm y) = \pm \frac{dy}{A \pm y}$ in secundo casu; *Quod erat* &c.

Corollaria.

1. Si ponatur $x^m = v$, erit $mlx = lv$; & differentiando fiet $m\frac{dx}{x} = \frac{dv}{v}$, seu $mv\frac{dx}{x} = dv$; at $v = x^m$; ergo $mx^{m-1}dx = dv$; quamobrem si statuatur $m = \frac{1}{n}$, erit $d.x^{\frac{1}{n}} = \frac{1}{n}x^{\frac{1}{n}-1}dx$.

2. Esto rectangulum $xy = v$. Sumptis logarithmis, fit $lx + ly = lv$, cujus differentiale $= \frac{dx}{x} + \frac{dy}{y} = \frac{dv}{v}$, sive $\frac{ydx + xdy}{xy} = \frac{dv}{v}$. Sed $xy = v$; ergo $d.(xy) = ydx + xdy$.

3. Generalius sit productum $x^m y^n = v$; erit $mlx + nly = lv$; cujus differentiale $m\frac{dx}{x} + \frac{ndy}{y} =$

dv

$\frac{dv}{v}$. seu $\frac{mydx + nxdy}{xy} = \frac{dv}{v}$; unde $v\frac{(mydx + nxdy)}{xy} = dv$; at $v = x^m y^n$; ergo $x^{m-1} y^{n-1} (mydx + nxdy) = my^n x^{m-1} dx + nx^m y^{n-1} dy = dv$. Verum ad hoc differentiale deveniri quoque potest per corollarium primum, si pro x substituatur x^m, & pro y ponatur y^n, quæ corollariis etiam subsequentibus sunt adplicanda.

4. Esto $xyz \ldots = v$. Ergo $lx + ly + lz \ldots = lv$, & $\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} + \ldots = \frac{dv}{v}$. Hinc $\frac{yzdx + xzdy + xydz + \ldots}{xyz \ldots} = \frac{dv}{v}$, ac tandem $yzdx + xzdy + xydz + \ldots = dv = d.(xyz \ldots)$.

5. Sit quotus $\frac{x}{y}$. Statue $\frac{x}{y} = v$. Ergo $lx - ly = lv$, ac proinde $\frac{dx}{x} - \frac{dy}{y} = \frac{dv}{v}$, seu $\frac{ydx - xdy}{xy} = \frac{dv}{v}$. Sed $x = vy$; Ergo $\frac{ydx - xdy}{vyy} = \frac{dv}{v}$, & $d.\left(\frac{x}{y}\right) = dv = \frac{ydx - xdy}{yy}$.

 6.

6. Sit quotus $\frac{xy\ldots\ldots}{tz\ldots\ldots} = v$. Ergo $lx + ly\ldots\ldots -$

$lt - lz\ldots\ldots = lv$, & $\frac{dx}{x} + \frac{dy}{y} + \ldots\ldots - \frac{dt}{t} - \frac{dz}{z} - \ldots\ldots$

$= \frac{dv}{v}$. Quare $\frac{tyzdx + txzdy + \ldots\ldots - xyzdt - txydz - \ldots\ldots}{txyz\ldots\ldots}$

$= \frac{dv}{v}$; Ex quo (multiplicando per $\frac{xy\ldots\ldots}{tz\ldots\ldots} = v$)

fit $\frac{tyzdx + txzdy + \ldots\ldots - xyzdt - txydz}{ttzz\ldots\ldots} - \ldots\ldots = dv$

$= d.\left(\frac{xy\ldots\ldots}{tz\ldots\ldots}\right)$.

7. Præterea si detur lx^m; ob $lx^m = mlx$, ejus differentiale erit $\frac{mdx}{x}$. Sed si fuerit $lx = v$, erit

$(lx)^m = v^m$; $d.(lx)^m = mv^{m-1}dv = m(lx)^{m-1}\frac{dx}{x}$.

8. Sit $a^x = v$; erit $xla = lv$; $ladx = \frac{dv}{v}$; $vladx = dv = a^x . ladx$.

9. Quoniam $l(xy) = lx + ly$; erit $\frac{dx}{x} + \frac{dy}{y} = d(lxy) = \frac{ydx + xdy}{xy}$

10.

10. Similiter $l\left(\frac{x}{y}\right) = lx - ly$; unde $d\left(l\frac{x}{y}\right)$

$= \frac{dx}{x} - \frac{dy}{y} = \frac{ydx - xdy}{xy}$.

11. Esto $x^y = v$; erit $ylx = lv$; hinc $y\frac{dx}{x} +$

$lxdy = \frac{dv}{v}$, sive $v\frac{(ydx + xlxdy)}{x} = dv = x^{y-1}ydx$

$+ x^y lxdy = d(x^y)$, &c.

Scholion I. Quum differentiale rectanguli xy sit per aliam methodum fundamentum totius Calculi Differentialis, libet istud sine differentialium secundi ordinis interventu sic quoque invenire.

Sint AP, vel Ap (*Fig.* 32. 33.) $= x$, PM, vel pm $= y$, & ambæ simul crescant, aut decrescant. Compleatur rectangulum PN, quo per rectam, vel curvam AM in duo triangula, vel in duo spatia curvilinea AMP, AMN diviso, lateribus PM, MN concipiantur ductæ infinite proximæ pm, mn recta vel curva AM interceptæ, quæ producantur usque ad occursum MN in S, & PM in o. Perspicuum est, spatiola $PMmp$, $MmnN$ esse duarum figurarum AMP, AMN differentialia, seu elementa; sed spatiola $PMSp$, $NSmn$ simul sumpta, seu $Pomp$, $MNno$ æquantur spatiolis $PMmp$, $NMmn$, & ita semper; ergo etiam spatiola $PMSp$, $NSmn$, seu $Pomp$, $MNno$ simul sumpta formant differentiale

 dua-

duarum figurarum AMP, AMN, seu rectanguli indeterminati PN; At $PMSp + NSmn$, sive $Pomp + MNno = ydx + xdy$; ergo differentiale rectanguli PN, seu producti xy est $ydx + xdy$.

Scholion II. Amplissimus in tota mathesi logarithmorum usus expeditiorem quoque reddit tangentium perquisitionem. Etenim exprimat Z functionem coordinatarum x, y; formula $y^m = Z$, vel $mly = lZ$ curvæ cujuslibet æquationem repræsentabit. Captis igitur differentiis, fiet $\frac{mdy}{y} = \frac{dZ}{Z}$, unde $mZ = \frac{ydZ}{dy}$. Itaque substitutis pro dZ in æquatione speciali differentialibus, & relicta in altero æquationis membro formula generali $\frac{ydx}{dy}$ subtangentem indicante, hujus valor in altero reperietur. Exemplo sit $y^3 = ax^2 - xy^2$. Erit $m = 3$, $Z = ax^2 - xy^2$, $dZ = 2axdx - y^2dx - 2xydy$; quare $3ax^2 - 3xy^2 = \frac{2axydx - y^3dx - 2xy^2dy}{dy}$, seu $3ax^2 - 3xy^2 + 2xy^2 = \frac{2axydx - y^3dx}{dy}$; Hinc $\frac{ydx}{dy} = \frac{3ax^2 - xy^2}{2ax - yy}$.

Scholion III. Fervet adhuc controversia, num dentur logarithmi quantitatum negativarum; libet hac occasione sequens proponere ratiocinium.

De-

Demonstravimus (*Introd.* §. 639. n. 6.), dari non posse $l-y=l+y$, quia foret $-y=+y$. Ergo nec dabitur $d.l-y=d.l+y$. Hoc præmisso, statue $-y=v$; erit $l-y=lv$, & $d.l-y=d.lv$

$$=\frac{dv}{v};\text{ restitutoque valore, fiet } d.l-y=-\frac{dy}{-y}$$

$$=\frac{dy}{y};\text{ sed } \frac{dy}{y}=d.l+y;\text{ ergo } d.l-y=d.l+y,$$

quod repugnat. Quum igitur obtineri nequeat differentiale quantitatis $l-y$ sine absurdi concursu, sequitur, logarithmum quoque quantitatis negativæ esse impossibile, quod pluribus modis non solum confirmatur, sed per ipsam expressionem $l-1$, quæ se statim denunciat imaginariam, evincitur, nam l

$-1=l\sqrt{(-1)^2}=2l\sqrt{-1}$, ut alibi (*ibid.* §. 643.) etiam monuimus. Ex quo colligimus, logythmicam, cui tam finitorum (*ibid.* §. 253.), quam infinite parvorum calculus innititur, non duobus ramis, ut nonnulli contendunt, sed unico esse præditam, quod celeberrimus Eulerus (*Introduct. in analys. infinit.* T. 2. pag. 290.) fuse demonstrat.

Propositio II. *Differentias primas* dx, dy, $\sqrt{dx^2+dy^2}$ *in quantitates finitas commutare.*

Resolutio. Sit curva quævis *AM* (*Fig.* 33 34), cujus ordinatæ infinite proximæ *PM*, *pm* (*Fig.* 33.), vel *PM*, *Pm* (*Fig.* 34.); si ex puncto *M* ducta *Mo* ordinatæ *pm*, vel *Pm* perpendiculari, & axi *AP* parallela, fiat $AP=x$, $PM=y$, subtangens *PT*

$= s$, tangens $MT = t$, erit $Mo = dx$; $om = dy$, $Mm = dt = \sqrt{dx^2 + dy^2}$. Jam vero quoniam habetur $dx : dy = s : y = \frac{s}{s} : \frac{y}{s} = 1 : \frac{y}{s}$, pariterque $dx : dt = s : t = 1 : \frac{t}{s}$, quotiescumque ponatur $dx = 1$, erit $dy = \frac{y}{s}$, & $\sqrt{dx^2 + dy^2} = \frac{t}{s}$. Idem sic etiam ostenditur; statue $dx = A$, stabit $dx : dy = A : A\frac{dy}{dx} = 1 : \frac{dy}{dx} = \frac{y}{s}$, &c. Similiter posita $dy = 1$, evadet $dx = \frac{s}{y}$, & $\sqrt{dx^2 + dy^2} = \frac{t}{y}$. Praeterea ob $dt : dx = t : s = 1 : \frac{s}{t}$, nec non ob $dt : dy = t : y = 1 : \frac{y}{t}$, facta perimetri curvilineæ differentia $dt = y$, obtinetur in terminis finitis differentia tam abscissæ, quam ordinatæ, adeoque elementi Mm valor tam per $\sqrt{1 + \frac{yy}{tt}}$, quam per $\sqrt{1 + \frac{ss}{tt}}$ designari poterit. Hujusmodi primæ differentiæ etiam per normales, & subnormales cum coor-

coordinatis, si libeat, exprimi possunt; *Quod erat* &c.

Coroll. In formulis igitur, quæ coordinatarum differentias primas involvunt, suffectis loco differentiarum supradictis valoribus, quantitates finitas formulæ repræsentabunt, ac proinde (1.°) formulæ istæ construi poterunt; (2.°) elicito valore s, subtangentes invenientur; (3.°) quum $\frac{dy}{dx} = 0$, vel $\frac{dy}{dx} = \infty$ maximis & minimis indagandis, ut notum est, inserviat, idem quoque præstabit $\frac{y}{s} = 0$, vel $\frac{y}{s} = \infty$. Hinc si concipiatur curva, cujus maxima vel minima ordinata exprimat maximum illud, vel minimum, quod quæritur, hoc habebitur, si ordinata per subtangentem dividatur, & quotus ponatur $= 0$, vel $= \infty$.

Exemplum I. Sit construenda formula $ydx + xdy$. Positis $dx = 1$, $dy = \frac{y}{s}$, formula evadet $y + \frac{xy}{s}$; itaque facta $PQ = \frac{xy}{s}$ (*Fig.* 33.), erit $ydx + xdy = QM$; quæ constructio si semper continuetur, exurget area QAM æqualis rectangulo $xy = PN$, curvæ AMP circumscripto, cujus differentia est data formula $ydx + xdy$, adeoque sublato communi spatio AMP, remanebit area AQP æqualis areæ AMN, unde *Theorema* Si

Si fiat ut subtangens PT *ad abscissam* AP, *ita ordinata* PM *ad quartam* PQ, *& ita semper, orietur novum spatium curvilineum* AQP *æquale spatio exteriori* AMN *datæ curvæ* AMP.

Hoc ita confirmatur: Quum sit $PT : PM = Mo : om = Pp : Nn = AP$ (vel MN) : PQ, fiet $PQ . Pp = MN . Nn$, & sic semper; quare tota area AQP toti areæ AMN æquabitur.

Scholion. Si quis aveat, ad omne scrupulum tollendum, differentias quoque secundi ordinis, velut Mmo, in computationem vocare, is aream AQP areæ AMN accuratissime æqualem semper reperiet; quamobrem quum spatii QAM differentiale sit $ydx + xdy$, eadem formula erit differentiale exactum rectanguli xy.

Exemplum II. Sit rursus construenda formula $ydx + xdy$, ponaturque $dy = 1$. Suffectis valoribus, ut supra, formula evadet $= s + x$; quare facillima erit constructio; facta enim (*Fig.* 35.) $NQ = PT = s$, quum sit $AP = x$, erit $QM = s + x$, proindeque mixtilineum $AQNMA$ æquabitur rectangulo $xy = PN$ genito a figuræ datæ AMP coordinatis; quapropter ablato spatio communi AMN, remanebit spatium $ANQ = AMP$, quod dat *Theorema*

Si fiat ut subtangens ad ordinatam, ita abscissa ad NQ *perpendiculariter collocatam, ut in figura, & ita semper, nascetur novum spatium curvilineum* ANQ *æquale spatio* AMP *curvæ datæ.*

Quod sic etiam demonstratur. Quum sit $PT : PM = Mo : om$, sive $QN : PM = Pp : Nn$, erit $QN . Nn = Pm . Pp$, & sic ubique; ergo &c.

Scho-

Scholion. Tam præsens, quam Theorema exempli antecedentis pro plurimis spatiis curvilineis dimetiendis sunt egregia, quod præter propositum animadverto.

Exemplum III. Construenda iterum occurrat dicta formula $ydx + xdy$, statuendo $\sqrt{dx^2 + dy^2} = 1$. Quoniam in tali casu habetur $dx = \frac{s}{t}$, & $dy = \frac{y}{t}$, valoribus hisce substitutis, prodit $ydx + xdy = \frac{sy + xv}{t}$. Itaque fiat $MT : TP = PM : MQ$ (*Fig.* 36.), quæ perpendiculariter insistat curvæ AM in puncto M ex parte exteriori, & ita semper; orietur figura AQM, cujus area areæ AMP æqualis erit. Etenim ob $MT : TP = Mm : Mo = PM : MQ$, erit $MQ . Mm = PM . Mo$, & hoc semper, propterea que tota area $AQM = AMP$; *quod erat primum.*

Fiat nunc $MT : PM = AP : MV$, quæ perpendiculariter insistat curvæ AM in puncto M ex parte anteriori, & ita semper; exurget figura AVM, cujus area æquabitur areæ AMN; quum enim sit $MT : PM = Mm : om = MN : MV$, erit $MV . Mm = MN . Nn$, quod $Nn = om$; quare tota area AMV toti areæ AMN æquabitur; *quod erat secundum.*

Jam vero areæ AMP, AMN rectangulo PN æquantur; ergo &c. *quod erat tertium.*

Exem-

Exemplum IV. Sit AMD (*Fig.* 37.) Cyclois vulgaris, cujus arculus genitor AEB; actis infinite proximis PM, pm circulum secantibus in punctis E, e, ex punctis M, E demittantur Mo, Er axi AP parallelæ, & tam ex centro C quam ex puncto E jungantur Ce, EA. Sit porro $AP = x$, $PM = y$, $AB = 2a$, arcus $AE = z$; erit $Mo = Er = dx$, $om = dy$, $Ee = dz$, $PE = \sqrt{2ax - xx}$. Quum autem ex natura cycloidis sit arcus $AE = EM$, ejus æquatio fiet $y = z + \sqrt{2ax - xx}$, sumptisque differentiis, erit $dy = dz + \frac{adx - xdx}{\sqrt{2ax - xx}}$. Sed ex triangulis similibus peC, Ere prodit $pe : Ce = Er : re$, nimirum $\sqrt{2ax - xx} : a = dx : \frac{adx}{\sqrt{2ax - xx}} = dz$; ergo factis debitis substitutionibus, & deleta dx, provenit $\frac{y}{s} = \frac{2a - x}{\sqrt{2ax - xx}}$, unde $s = y\,\frac{\sqrt{2ax - xx}}{2a - x} = \frac{y\sqrt{x}}{\sqrt{2a - x}} = \frac{xy}{\sqrt{2ax - xx}}$.

Hinc ob $\sqrt{2ax - xx} : x = y : s$, sive $PE : AP = PM : PT$, patet, tangentem in cycloide esse chordæ respondenti AE circuli genitoris parallelam.

Scholion. Datæ curvæ subnormalem, adeoque subtangentem, etiam sine trianguli Barrowiani subsidio pos-

possumus determinare. Etenim finge parabolam apollonianam DME (*Fig.* 38.) concavam versus axem DP, quæ tangat, sive osculetur per laterculum Mm datam curvam AMB, cujus axis ADP. Ordinatis infinite proximis PM, pm, & parabolæ ducta normali SM, erit ex Conicis PS æqualis dimidio parametri ejusdem parabolæ. Ponatur modo $AP = x$, $DP = z$, $PM = y$, PS

$= \frac{r}{2}$, erit $Pp = dx = dz$. Quoniam vero PM est

utrique curvæ AM, DM communis, (ut sunt quoque subnormalis, normalis, subtangens, ac tangens) quantitates in utraque curva, quæ ordinatæ y ad eandem potestatem assurgenti æquantur, æquabuntur inter se; facta igitur harum æquatione, sumptisque differentiis, si dx, dz, quippe æquales, deleantur, nova prodibit æquatio in terminis finitis, quæ altera sui parte quantitatem r continebit, quapropter si ita ordinetur æquatio, ut ex altero mem-

bro $\frac{1}{2} r$ sola supersit, ex altero subnormalis quæsitæ valor emerget. Sic si fuerit $2ax \mp xx = yy$, (nempe cum signo negativo ad circulum, cum affirmativo ad hyperbolam æquilateram), quum æquatio ad parabolam sit $yy = rz$, erit $2ax \mp xx = rz$, quare $2adx \mp 2xdx = rdz$, sive $2a \mp 2x = r$, adeoque

$\frac{1}{2} r = a \mp x = PS$.

Pro

Pro definienda vero subtangente expeditius quam fert methodus consueta, revocato triangulo Barrowiano, sit logystica AMB, cujus asymptotus TE (*Fig.* 39.); ordinatis PM, pm infinite propinquis, & ex M demissa Mo asymptoto TE parallela, imaginare datam curvam DMF, quæ cum priori per laterculum Mm congruat. Quum subtangens PT, quæ in logystica est constans, sit utrique curvæ communis, positis quoad logysticam $Pp = dz$, & quoad curvam datam $DP = x$, erit etiam $Pp = dx = dz$. Itaque factis $PT = s$, $PM = y$, & $mo = dy$, quum sit $dy : dz = y : s$, adeoque $dy = \frac{ydz}{s}$, sumptis in æquatione ad curvam datam differentiis, & pro dy substituendo $\frac{ydz}{s}$, si in nova æquatione deleantur æquales dz, dx, æquatio superstes in terminis finitis apparebit, ex qua valor s facillime elicietur. Sic si fuerit $y^3 - x^3 = axy$; differentiando, erit $3y^2dy - 3x^2dx = axdy + aydx$, sive $\frac{3y^3dz - axydz}{s} = aydx + 3x^2dx$, sive $\frac{3y^3 - axy}{s} = ay + 3x^2$; ex quo prodit $s = \frac{3y^3 - axy}{ay + 3x^2}$.

Similiter ope Tractoriæ valores tangentium, & ope circuli valores normalium investigantur.

Pro-

Propositio III. *Differentias altiorum graduum in quantitates finitas transformare.*

Resolutio. Sit curva quælibet AM (*Fig.* 40.), cujus abscissa $AP = x$, ordinatæ infinite proximæ PM, $pm = y$, subtangens $PT = s$; erit $Pp = Mo = dx$, $om = dy$. At in *Fig.* 41., in qua ordinatæ PM, $Pm = y$ exeunt a puncto fixo P, centro P, radio PM descripto arculo Mo, fiat hic $= dx$, retentis ut ante $om = dy$, $PT = s$.

Jam vero si adsumatur $dx = 1$, erit $dx^n = 1$, $ddx = 0$, nec non $d^n x = 0$; $dy = \frac{y}{s}$, & $dy^n = \frac{y^n}{s^n}$. Restat igitur, ut differentias ddy, $dddy$, &c. in quantitates finitas convertamus. Quum itaque sit $dy = \frac{y}{s}$, captis iterum differentiis, erit $ddy = \frac{sdy - yds}{s^2} = \frac{y - yds}{s^2}$. Quare (postquam in *Fig.* 41. ductus sit axis AP, eique ordinata MV) si in MP (*Fig.* 40.) vel VM (*Fig.* 41.) protracta fiat $PN = PT$, & ita semper, orietur curva ANP, cujus subtangens PR (ordinatæ PN semper normalis in P) ponatur $= f$, ordinatisque infinite proximis PN, pn (*Fig.* 40.) vel Pn, Pn (*Fig.* 41.), nec non sumpto arculo Nr = arculo Mo, erit $nr = ds = \frac{PN}{PR} = \frac{s}{f}$, ac

proin-

proinde $ddy = \frac{y - ds}{s^2} = \frac{yf - ys}{fs^2} = \frac{y(f - s)}{fs^2} =$

$\frac{PM(PR - PT)}{PR \cdot PT^2}$.

Deveniamus modo ad differentiale tertii ordinis $dddy$. Hoc ut in quantitates finitas transformetur, facta $PQ = PR = f$, & ita semper, exurget figura AQP, cujus subtangente PK posita $= v$, erit differentia $qe = df = \frac{PQ}{PK} = \frac{f}{v}$, nam in *Fig.* 41. arculus Qe semper æqualis arculo Mo consideratur; deinde differentiale secundi ordinis $\frac{y(f - s)}{fs^2}$, vel potius $\frac{y - yds}{s^2} = ddy$ rursus differentietur, substituendo $\frac{y}{s}$ pro dy, eritque $d^3y = \frac{y - ysdds - 3yds + 2yds^2}{s^3}$;

sed $ds = \frac{s}{f}$; ergo $dds = \frac{fds - sdf}{f^2} = \frac{s - sdf}{f^2}$;

sed $df = \frac{f}{v}$; ergo $dds = \frac{sv - sf}{vf^2}$; quibus valoribus suffectis, habetur $dddy =$

$\left(y - \frac{ys^2(v - f)}{vf^2} - \frac{3ys}{f} + \frac{2ys^2}{f^2}\right) : s^3 =$

vy

$$\frac{vy(f^2+s^2)-fsy(3v-s)}{vf^2s^2} =$$

$$\frac{PK.PM(\overline{PR}^2+\overline{PT}^2)-PT.PR.PM(3PK-PT)}{PK.\overline{PR}^2.\overline{PT}^3}.$$

Eadem methodo procedendum in differentiis ordinum eminentiorum ad quantitates finitas revocandis. *Quod erat* &c.

Corollaria.

1. Si formulæ dx, ddx, $dddx$, &c. sint ad quantitates finitas redigendæ, facta $dy = 1$, similis erit operatio.

2. Pro formulæ $d^n y$, vel $d^n x$ reductione tot curvæ requiruntur, quot unitates numerus integer affirmativus n comprehendit.

3. Si problema aliquod generale solvatur per formulam, quam coordinatarum differentiæ ad quorvis gradus elevatæ ingrediantur, problematis hujus solutio generali quoque constructione donabitur.

4. Differentiales cujuscumque gradus ad differentiales primi gradus, si lubeat, deprimi possunt, ut in Exemplis sequentibus.

Exemplum I. Esto $y^m = a^{m-1}x$ ad infinitas parabolas. Quoniam $s = mx$, posita $dx = 1$, erit $dy = \frac{y}{mx} = \frac{1}{m} a^{m-1} y^{1-m}$; $ddy = \frac{(1-m)}{m} a^{m-1} y^{-m} dy$

$= \frac{(1-m)}{m^2} a^{2m-2} y^{1-2m}$; $dddy =$

$\frac{(1-m)(1-2m)}{m^2} a^{2m-2} y^{-2m} \; dy =$

$\frac{(1-m)(1-2m)}{m^3} a^{3m-3} y^{1-3m}$; $ddddy =$

$\frac{(1-m)(1-2m)(1-3m)}{m^3} a^{3m-3} y^{-3m} \; dy =$

$\frac{(1-m)(1-2m)(1-3m)}{m^4} a^{4m-4} y^{1-4m}$; &c.

ubi apparet differentiarum altiorum progreſſio in infinitum.

Exemplum II. Eſto formula $\frac{(dx^2 + dy^2)\sqrt{dx^2+dy^2}}{-dxddy}$

Poſita $dx = 1$, quum ſit $dx^2 + dy^2 = 1 + \frac{y^2}{s^2} =$ $\frac{s^2+y^2}{s^2}$, & $-ddy = \frac{yds - y}{s}$, valoribus hiſce in allata formula ſubrogatis elicietur

$\frac{(dx^2+dy^2)\sqrt{dx^2 + dy^2}}{-dxddy} = \frac{(s^2+y^2)\sqrt{s^2 + y^2}}{(ds-1)sy} =$

$\frac{\overline{MT}^3 . PR}{PT . PM(PT - PR)}$ (*Fig.* 40. 41.) propter ds

$=$

$= \frac{PT}{PR}$; sed per triangula similia PTM, TMS, habemus $PT : TM = TM : TS = \frac{\overline{TM}^2}{PT}$; ergo substituendo, dicta formula evadet $\frac{MT \cdot ST \cdot PR}{PM(PT - PR)}$; vel extensa normali SM ultra curvam, donec occurrat in L rectæ TL perpendiculari in T subtangenti PT, quum ex triangulis similibus PMT, TLS fiat $LS = \frac{TM \cdot TS}{PM}$, rursus substituendo, tandem prodibit $\frac{(dx^2 + dy^2)\sqrt{dx^2 + dy^2}}{-dxddy} = \frac{LS \cdot PR}{PT - PR}$.

Hæc dictæ formulæ reductio etiam obtinetur, si statuatur $dy = 1$. Sed quum ipsa formula radii osculatoris doctrinam in mentem revocet, haud inutile censeo, methodum consueta breviorem pro ipso radio definiendo exponere.

Propositio IV. *Data curva evolutionis* AM (*Fig.* 42.), *cujus ordinatæ* PM *sint axi* AP *perpendiculares*, *cujusque tangens* MT *innotescat*, *radii osculatoris* MC *longitudinem determinare.*

Resolutio. Sumpto in perimetro AM elemento Mm, ex punctis M, m ductis radiis osculatoribus MC, mC, tangentibus MT, mH, usque ad occursum protractæ PA, ordinatis MP, mp, nec non centro M, radio MT descripto arculo TI, ponatur

tur $AP = x$, $PM = y$, $PT = s$, $MT = t$; erit $Pp = Mo = dx$, $mo = dy$, TH vero (differentiale rectæ $AT = PT - AP = s - x$) $= ds - dx$. Triangula similia HTI, MTP dant $TI = y\frac{(ds - dx)}{t}$, hinc (ob triangula similia CMm, Mnm) ex sectoribus similibus CMm, mTI eruitur $MC = \frac{t^2\sqrt{dx^2 + dy^2}}{y(ds - dx)}$; *Quod erat* &c.

Coroll. 1. Si statuatur $dx = 1$, erit $dy = \frac{y}{s}$; quare substituendo fit $MC = \frac{t^3}{sy(ds - 1)}$. Si $dy = 1$, erit $dx = \frac{s}{y}$, adeoque $MC = \frac{t^3}{y(yds - s)}$.

Coroll. 2. Si subnormalem in computationem tradere magis arrideat, posita subnormali $PS = r$, erit $s = \frac{y^2}{r}$; $s^2 = \frac{y^4}{r^2}$; $ds = \frac{2rydy - y^2dr}{rr}$; $t^2 = s^2 + y^2 = \frac{y^4 + r^2y^2}{rr}$; quapropter factis in formula generali $\frac{t^2\sqrt{dx^2 + dy^2}}{y(ds - dx)}$ substitutionibus, proveniet $\frac{(y^3 + r^2y)\sqrt{dx^2 + dy^2}}{2rydy - y^2dr - r^2dx}$; posita vero $dx = 1$, fit $dy =$

$= \frac{r}{y}$; ergo substituendo, emergit $\frac{(y^2 + r^2)^{\frac{3}{2}}}{r^2 - y^2 dr} = MC$.

Quod si statuatur $dy = 1$, quoniam $dx = \frac{y}{r}$, erit $MC = \frac{(y^2 + r^2)^{\frac{3}{2}}}{r(r - ydr)}$.

Coroll. 3. Analogia $MT : PT = MC : MQ$ dat $t : s = \frac{t^2 \sqrt{dx^2 + dy^2}}{y(ds - dx)} : \frac{st\sqrt{dx^2 + dy^2}}{y(ds - dx)}$, ita ut si fiat $dx = 1$, erit $dy = \frac{y}{s}$; ac proinde coradius $MQ = \frac{t^2}{y(ds - 1)}$, &c.

Exemplum I. Esto $y^m = x$ ad infinitas parabolas (*Fig.* 42.), si m est numerus affirmans, ad infinitas hyperbolas intra asymptotos (*Fig.* 43.), si negans; erit $s = mx$, $ds = mdx = m$, posita $dx = 1$; quare adhibita coradii formula $\frac{t^2}{y(ds - 1)}$, erit $MQ = \frac{t^2}{(m - 1)y}$, & $MC = \frac{t^3}{sy(ds - 1)} = \frac{t^3}{(m - 1)sy}$.

Constructio pro parabolis. Ex subtangentis PT puncto extremo T excitetur perpendicularis TL occurrens in L normali SM extra curvam datam AM productæ; hæc rectam LS determinabit, cujus mul-

tiplum vel submultiplum erit radius osculator generalis quaesitus. Quum enim sit $MC = \frac{\overline{MT}^3}{(m-1)PT.PM}$, ob $\frac{\overline{MT}^2}{PT} = TS$, erit $MC = \frac{MT.TS}{(m-1)PM}$. Sed per triangula similia TMP, TLS, stat $PM : MT = TS : LS = \frac{MT.TS}{PM}$; ergo $MC = \frac{LS}{m-1}$.

Quoniam vero radii osculatoris formula generalis consueta est $\frac{(dx^2+dy^2)^{\frac{3}{2}}}{-dxddy}$, ex hujus jam exhibita constructione, sive ad quantitates finitas reductione $\frac{PR.LS}{PT-PR}$ idem radii MC valor statim elicitur. Aequatio enim ad figuram ANP (*Fig.* 40.) sit $PN = PT = mx = mAP$, adeoque quum ANP sit triangulum erit $PR = AP$; hinc suffectis valoribus, formula $\frac{PR.LS}{PT-PR}$ evadit $\frac{AP.LS}{mAP-AP} = \frac{LS}{m-1} = MC$.

Constructio pro hyperbolis OR *intra asymptotos* AT, AV. Ex termino T tangentis MT (*Fig.* 43.) excitata ut supra perpendiculari TQ occurrente in Q ordinatae MP productae, ex puncto Q agatur QL axi

axi AP parallela secans in L protractam normalem MS; erit $\frac{LM}{m+1} = MC$. Etenim $\frac{-s^3}{(m+1)sy} =$

$$\frac{-1}{m+1}\left(\frac{s\sqrt{s^2+y^2}}{y} + \frac{y\sqrt{s^2+y^2}}{s}\right) = \frac{-1}{m+1}$$

$$\left(\frac{IT.MT}{PM} + \frac{PM.MT}{PT}\right) = \frac{-1}{m+1}(TQ+SM)$$

$$= \frac{-1}{m+1}LM.$$

Vel sic: Quum sit $MQ = \frac{\overline{MT}^2}{PM}$; & $ML =$

$\frac{MQ.MT}{PT}$; erit $\frac{\overline{MT}^3}{(m+1)PT.PM} = \frac{-s^3}{(m+1)sy}$

$$= \frac{-LM}{m+1}.$$

Idem quoque per methodum generalem sic evincitur. Per Prop. III. Exemplum II. habemus $\frac{(dx^2+dy^2)^{\frac{3}{2}}}{-dxddy} = \frac{(s^2+y^2)^{\frac{3}{2}}}{(ds-1)sy} = \frac{MT.TS.PR}{PM(PT-PR)}$; sed extensa normali MS (*Fig.* 43.) citra curvam, donec occurrat in L rectæ QL, quum per triangula similia PMT; QLM sit $\frac{TM.TS}{PM} = LM$; ergo

ſubſtituendo, emerget $\frac{LM.PR}{PT-PR} = \frac{(s^2+y^2)^{\frac{3}{2}}}{(ds-1)sy}$;

quare quoniam figura *ANP* (*Fig.* 40.) eſt triangulum, in quo $PN = PT = -mAP$; ſuffectis valoribus, ob $PR = AP$, fiet $\frac{LM.PR}{PT-PR} = \frac{LM}{-n-1}$
$= -\frac{LM}{m+1}$.

Proprietates, quæis præditæ ſunt parabolarum & hyperbolarum radii oſculatores reticeo, quod ubique reperiantur expoſitæ, hoc unum addo, duas evolutas a radiis oſculatoribus tangendo genitas in qualibet hyperbola aſymptotica, præter apollonianam, eſſe naturæ prorſus diverſæ, ac proinde affirmari non poſſe, eſſe eandem curvam flexu contrario donatam.

Exemplum II. Sit æquatio $y^2 = \frac{abx \mp bx^2}{a}$ cum ſigno negativo ad ellipſim, cum affirmativo ad hyperbolam ad axem relatam. Subnormalis r valor $=$ $\frac{1}{2}b \mp \frac{bx}{a}$; hujus differentia (poſita $dx = 1$) eſt $\mp$ $\frac{b}{a}$, quo valore in formula $\frac{(r^2+y^2)^{\frac{3}{2}}}{r^2 - y^2 dr}$ ſubrogato, prodit $\frac{a(r^2+y^2)^{\frac{3}{2}}}{ar^2 \pm by^2}$; ſed $ar^2 \pm by^2 = \frac{1}{4}ab^2$; ergo

a

$\frac{a(r^2 + y^2)^{\frac{3}{2}}}{ar^2 \pm by^2} = \frac{4(r^2 + y^2)^{\frac{3}{2}}}{b^2}$. Hinc in ellipsi, & hyperbola si ad parametrum, & normalem quarta continue proportionalis quæratur, ejus quadruplum erit radius osculator.

Exemplum III. Esto Tractoria, in qua tangens est constans, sive $= a$; ergo subtangens $s = \sqrt{a^2 - y^2}$; quapropter posita $dx = 1$, erit $ds = -\frac{ydy}{\sqrt{a^2 - y^2}}$

$= -\frac{y^2}{a^2 - y^2}$. Itaque $\frac{(s^2 + y^2)^{\frac{3}{2}}}{sy(ds - 1)} = -\frac{a\sqrt{a^2 - y^2}}{y}$

$= -\frac{as}{y}$; videlicet radius osculi in tractoria est quarta proportionalis post ordinatam, tangentem, & subtangentem, sumenda in plaga opposita, quod curva est axem versus convexa.

Exemplum IV. In logystica ob $s = a$, & $ds = 0$, habetur illico radius osculator $= \frac{(a^2 + y^2)^{\frac{3}{2}}}{-ay}$, quartus nempe proportionalis post ordinatam, tangentem, & aggregatum ex subtangente, & subnormali.

Lemma. *In quavis curva* AM (Fig. 44.), *cujus ordinatæ* PM *a puncto fixo, seu foco* P *procedunt, sumpta perimetri particula infinitesima* Mm, *ductisque tangentibus* MT, mH *occurrentibus subtangenti*

ti PH *perpendiculari in* P *recta* PM, *intercepta* HT *valorem determinare.*

Resolutio. Juncta Pm; eidem excitetur in P perpendicularis PL occurrens rectæ mH in L; nec non centro P, intervallis PT, PL, PM describantur arculi TK, LG, Mo, sitque $PM = y$, $Mo = dx$, $PL = s$; erit $KL = TG = ds$. Præterea sectores similes PMo, PLG præbent $GL = \frac{sdx}{y}$; sed triangula similia MPT, GLH dant $y : s = \frac{sdx}{y} : \frac{ssdx}{yy} = GH$; ergo $TH = ds + \frac{s^2 dx}{y^2} = \frac{y^2 ds + s^2 dx}{y^2}$; *Quod erat* &c.

Propositio V. *Data curva qualibet* AM (*Fig.* 44.), *cujus ordinatæ* PM *a puncto fixo; seu foco; seu polo* P *exeunt; ejusque nota sit tangens* MT *occurrens perpendiculari* PT *supra* PM, *formulam generalem pro radio osculatore* MC *invenire.*

Resolutio. Iisdem manentibus, ut in lemm. antec. centroque M descripto arculo TI, ponatur $MT = t$. Quoniam triangula similia PTM, TIH præbent $t : y = \frac{y^2 ds + s^2 dx}{y^2} : \frac{y^2 ds + s^2 dx}{ty} = TI$, sectores similes MTI, CMm dabunt $\frac{y^2 ds + s^2 dx}{ty} : t$

=

$= \sqrt{dx^2 + dy^2} : \frac{r^2 y \sqrt{dx^2 + dy^2}}{y^2 ds + s^2 dx} = MC$. *Quod erat* &c.

Coroll. 1. Posita $dx = 1$, quum sit $dy = \frac{y}{s}$, substituendo, erit $MC = \frac{yr^3}{s(y^2 ds + s^2)}$. Si fiat $dy = 1$, ob $dx = \frac{s}{y}$, evadet $MC = \frac{yr^3}{y^3 ds + s^3}$.

Coroll. 2. Si pro subtangente subnormalem in formula inventa introducere lubeat, fiat subnormalis $PS = r$, erit $s = \frac{y^2}{r}$; $s^2 = \frac{y^4}{r^2}$; $ds = \frac{2rydy - y^2 dr}{r^2}$; $r^2 = s^2 + y^2 = \frac{y^4 + r^2 y^2}{r^2}$, quibus valoribus in formula $\frac{yr^2 \sqrt{dx^2 + dy^2}}{y^2 ds + s^2 dx}$ subrogatis, hæc migrabit in aliam $\frac{(y^2 + r^2) \sqrt{dx^2 + dy^2}}{2rdy - ydr + ydx}$, ita ut si calculi leniendi causa ponatur $dx = 1$, erit $dy = \frac{r}{y}$, & formula novissima evadet $\frac{(y^2 + r^2)^{\frac{3}{2}}}{2r^2 - y^2 dr + y^2}$; si vero statuatur $dy = 1$, erit $dx = \frac{y}{r}$, & eadem formu-

mula fiet $\frac{(y^2+r^2)^{\frac{3}{2}}}{2r^2-rydr+y^2}$; quarum formularum quæ ad problematis solutionem expeditior experietur, est reliquis præferenda.

Coroll. 3. Demissa ex puncto C supra ordinatam MP productam perpendiculari CQ, quum triangula similia Mmo, MCQ dent $Mm : Mo = MC : MQ$, sive

$$\sqrt{dx^2+dy^2} : dx = \frac{yt^2\sqrt{dx^2+dy^2}}{y^2ds+s^2dx} : \frac{yt^2dx}{y^2ds+s^2dx},$$

si ponatur $dx = 1$, subosculatrix vel coradius MQ fiet $= \frac{yt^2}{y^2ds+s^2}$; vel quum sit $\sqrt{dx^2+dy^2} : dx$

$$= \frac{(y^2+r^2)\sqrt{dx^2+dy^2}}{2rdy-ydr+ydx} : \frac{dx(y^2+r^2)}{2rdy-ydr+ydx} =$$

MQ, posita $dx = 1$, evadet $MQ =$

$$\frac{(y^2+r^2)}{2rdy-ydr+y} = \frac{y(y^2+r^2)}{2r^2-y^2dr+y^2}.$$

Coroll. 4. Quoniam $s^2 = t^2 - y^2$, formula generalis radii osculatoris evadit $\frac{yt^2\sqrt{dx^2+dy^2}}{y^2ds+t^2dx-y^2dx}$; quapropter si y ponatur infinita, ut in casu Prop. antecedentis, quum termini, in quibus illa non reperitur, præ reliquis evanescant, ideoque deleri possint, formula fiet $\frac{t^2\sqrt{dx^2+dy^2}}{(ds-dx)}$, prorsus ut supra.

Exem-

Exemplum I. Sit AMB (*Fig.* 45.) spiralis hyperbolica, in qua subtangens $AT = a$ est constans. Fiat ordinata $AM = y$, arculus Mo centro A descriptus, & ordinatam infinite proximam Am secans in o, $= dx = 1$; erit $mo = dy$, & æquatio ad curvam $ady = ydx$. Quum sit $s = a$, fiet $ds = 0$; quare coradius $\frac{yt^2}{y^2 ds + s^2} = \frac{y(a^2 + y^2)}{a^2}$, & radius $= \frac{y(a^2 + y^2)^{\frac{3}{2}}}{a^3}$.

Constructio. Ex subnormalis AS puncto S ducta SQ parallela tangenti MT, occurrens in Q ordinatæ MA productæ, abscindet $MQ = \frac{y(a^2 + y^2)}{a^2}$; quum enim sit $AS = \frac{y^2}{a}$, erit $TS = \frac{a^2 + y^2}{a}$; sed triangula similia TMA, ASQ dant $TA : TS = MA : MQ = \frac{y(a^2 + y^2)}{a^2}$; ergo MQ erit coradius, ac proinde ducta QC subnormali AS parallela secans in C protractam MS, radius osculator MC determinabitur.

Exemplum II. Esto Logarithmica spiralis AMB (*Fig.* 45.), ejus naturæ, ut ducta ad quodvis punctum M tangente MT, & ex foco A ordinata AM, angulus AMT sit ubique constans. Fiant omnia ut in Exem-

Exemplo antecedente. Quum ratio mo ad oM sit pariter constans, nimirum ut a ad b, erit subtangens $AT = s = \frac{by}{a}$, $ds = \frac{bdy}{a} = \frac{by}{as} = 1$; quare coradius $= \frac{ys^2}{s^2} = y = MA$, radius $= \frac{y\sqrt{a^2+b^2}}{b} =$ normali MS.

Corollaria.

1. Quoniam triangula TMS, MSA sunt similia, anguli TMA, ASM erunt æquales; sed angulus TMA semper est constans, ergo semper constans erit etiam angulus ASM; quare quum tam MT, quam SM sint tangentes, evoluta AS erit eadem numero logarithmica spiralis AMB, sed situ inverso disposita.

2. Hinc habetur hujus curvæ rectificatio; est enim portio AS æqualis suæ tangenti SM resectæ a normali AM ex ejus foco, seu centro A prodeunte.

3. Quinimo dabitur & ejusdem curvæ quadratura; nam extensa curvedine AEM in rectam MT, evadent AM, Am ordinatim applicatæ triangulo rectangulo TMA, quod ob angulum AmM constantem erunt MA, mA parallelæ, quodque sunt in ratione constanti cum perimetrorum curvilinearum portionibus respondentibus; sed omnia minima rectanctula triangulum MTA componentia sunt dupla respondentium triangulorum AMm spatium logarithmicum $MAEM$ componentium; ergo triangu-

gu-

gulum rectangulum MTA est duplum spatii $MAEM$; sive spatium $MAEM$ logarithmicæ spiralis AEM est dimidium trianguli MTA rectanguli in A, cujus latera ordinata AM, tangens MT, & subtangens AT constituunt. Ex quo patet, logarithmicam spiralem esse triangulum rectangulum convolutum.

4. Si fuerit $mo = oM$, evadet $AM = AT$, ac proinde $SM = MT$; quare portio AS tunc erit ut supra similis & æqualis portioni respondenti AEM, & utraque curva AS, AM logarithmica spiralis æquilatera appellari poterit.

Scholion. Ante quam huic additamento finem imponam, sequentem addere mihi liceat animadversionem.

Si circulo ADR (*Fig.* 46.) filum sit pluries ad arbitrium advolutum, ita ut ex parte excedenti AB, quæcumque ea sit, sese explicando, evolutam BIK describat; hujus curvæ elementa BG, GH, HI, &c. in progressione arithmetica procedent, eruntque radiis osculatoribus proportionalia.

Etenim resoluta circumferentia in sua æqualia elementa AE, EF, &c. ductis radiis AC, EC, FC, &c. tangentibusque AB, EG, FH, &c. quæ curvæ BIK erunt radii osculatores, quum anguli BEG, GFH, &c. sint omnes æquales, omnes sectores BEG, GFH, &c. erunt similes; hinc $BG : GH = BE : GF = AB + 1 : AB + 2$, posita $AE = 1$, & sic deinceps, donec filorum evolutio perseverabit. Quod si filum circumferentiæ circumvolutum evolvi sine ulla excedenti portione AB concipiatur, curvam aliam ALM generabit a priori diversam, cujus ele-

men-

menta vi ejusdem ratiocinii erunt & ipsa in progressione arithmetica instar numerorum naturalium 1, 2, 3, 4, 5, &c. Unde profluit solutio Problematis

Innumeras curvas invenire, in quibus perimetrorum elementa in serie arithmetica progrediantur.

AD-

ADDITAMENTUM II.

IN Libro *de Maximis*, *&* *Minimis ad Institutiones geometricas accommodatis*, quem anno 1774. publici juris feci, ad casum particularem in Coroll. 1. Theor. III. Lib. I. pag. 4., n. 8. me restrinxi, neque generalem, ut debueram, solutionem, alibi fortasse distractus, præbui; ut quod ibi deest hic expleam, resumpta ejusdem libri *Fig.* 1. age datæ AB (*Fig.* 47.) parallelam DK, cujus ad occursum in K protracta CE, junge LK, IK, LD. Tres casus dari possunt, nimirum vel LE est *minor* quam LK, vel *major*, vel *æqualis*. Si *minor*; valet exhibita demonstratio in Libro cit. Si *major*; quum sit $LE > LK = ID$ (*per constr.*) & $LD = IK > IE$, erit $LE + LD > EI + ID$. Si *æqualis*; ob $LE = LK = ID$, & $LD = IK > EI$, erit rursus $EL + LD > EI + ID$. Ergo quoniam ex duobus punctis D, E inclinatæ duæ rectæ ad punctum C, ubi anguli BCE, ACD evadunt æquales, *minimum* aggregatum $EC + CD$ efficiunt; si (1.°) accipiantur duo puncta L, I a puncto C æquidistantia, junganturque EL, LD, EI, ID, *minimum* aggregatum secundarium erit ex parte, ubi juncta DE a recta AB positione data divergit; (2.°) Si ducta recta quavis terminata HD secante dictam AB in puncto aliquo C, sumantur hinc inde a puncto C duo puncta æquidistantia I, L, junctisque HI, ID, HL, LD, sit $HC < CD$;

ob $CD > CE = CH$; erit semper ED divergens ex parte D a data AB, adeoque $HI + ID < HL + LD$, & angulus HID major angulo HLD ob basim communem, & æquales altitudines triangulorum HID, HLD. Hac occasione pro alio ejusdem Libri supplemento haud inelegans, neque per solam geometriam multum facile, mihi fas sit demonstrare Theorema; sed prius sit

Lemma. *Conus rectus* ABC (Fig. 48.) *concipiatur sectus per plana* KL, GI *basi parallela, ita ut circuli a sectionibus hisce progeniti efficiant cum baseos circulo proportionem arithmeticam; ajo,* EF *ad* FD [*axis interceptas*] *minorem habere rationem, quam circulus, cujus diameter* AB, *ad circulum, cujus diameter* GI, *adeoque fore* $FE . \overline{GI}^2 < FD . \overline{AB}^2$.

Dem. Centro C, radiis CE, CF, CD describantur circuli, quorum diametri EN, FM, DH. Trium horum circulorum differentiæ erunt [*ex hyp.*] æquales, ac proinde æqualia erunt rectangula FEM, DFH. Hinc $EF : FD = FH : EM$; sed $FH : EM < FH + FD : EM - MN$; ergo $EF : FD < DH : EN$, & potiori jure $EF : FD < \overline{DH}^2 : \overline{EN}^2$. At $\overline{DH}^2 : \overline{EN}^2 = \overline{AB}^2 : \overline{GI}^2$; ergo $EF : FD < \overline{AB}^2 : \overline{GI}^2$; unde $EF . \overline{GI}^2 < FD . \overline{AB}^2$. *Quod erat* &c.

Coroll. Quin etiam ob $EF : FD < DH : EN$, multo magis erit generatim $EF : FD < \overline{DH}^n : \overline{EN}^n$, po-

posito n numero integro affirmativo; quare $EG . \overline{GI}^n < FD . \overline{AB}^n$.

Theorema. *Sint duæ figuræ inæquales, nec non similes tam invicem, quam cum alia figura, hac conditione, ut aggregatum basium in duabus primis figuris inæqualibus adæquet duplum baseos tertiæ figuræ; dico, duplum hujus tertiæ figuræ constituere præ reliquis simul sumptis minimum aggregatum.*

Dem. Si figuræ sint superficies expressæ per triangula similia CEI, CFL, CDB, quorum bases, vel perimetri, vel latera homologa sint in progressione arithmetica, quoniam propter $CF . EI + CF . DB = 2CF . FL$, fit $CF . EI + CF . DB < CE . EI + CD . DB$ (plus enim additur majori producto $CF . DB$, quam detrahitur e minori $CF . EI$), proveniet $2CF . FL < CE . EI + CD . DB$; *Quod erat primum.*

Solida, de quibus est quæstio, quorumque bases vel superficies similes, vel quadrata laterum homologorum in proportione arithmetica progrediuntur, exprimantur per tres cylindros, vel per tria parallelepipeda, quorum bases $\overline{AB}^2$, $\overline{KL}^2$, $\overline{GI}^2$, altitudines CD, CF, CE sumptæ in cono recto ABC. Patet (*ex hyp.*) esse $2\overline{KL}^2 . CF = \overline{AB}^2 . CF + \overline{GI}^2 . CF$; sed (*per Lemma*) $\overline{AB}^2 . CF + \overline{GI}^2 . CF$ <

$< \overline{AB}^2 . CF + \overline{AB}^2 . FD + \overline{GI}^2 . CF - \overline{GI}^2 . EF$
$= \overline{AB}^2 . CD + \overline{GI}^2 . CE$; ergo $2\overline{KL}^2 . CF <$
$\overline{AB}^2 . CD + \overline{GI}^2 . CE$; *Quod erat secundum.*

Corollaria.

1. Si ponatur $CD = AB$, ſtabit $2\overline{KL}^3 < \overline{AB}^3 + \overline{GI}^3$. Atqui omnia ſolida ſimilia cujuslibet configurationis, regularia, vel irregularia, comparantur per cubos laterum homologorum; ergo manentibus recenſitis conditionibus, idem de univerſis ſolidis ſimilibus, quod de cubis demonſtravimus, eſt ſtatuendum. Neque aliter de figuris planis ſimilibus rectilineis, vel curvilineis, vel mixtilineis, in quibus per quadrata laterum homologorum fit comparatio, eſt ratiocinandum.

2. Viciſſim ſi planum aliquod, vel aliquod ſolidum diſpeſci debeat in duo plana, vel in duo ſolida ſimilia, idque omnibus modis poſſibilibus, quodcumque ſit figurarum planarum, vel ſolidarum genus; par illud, quod duobus planis, vel duobus ſolidis invicem æqualibus conſtabit, *maximum* perimetrorum in primo caſu, ſuperficierum in ſecundo, immo partium analogarum ejuſdem numeri aggregatum exhibebit. Sic rectæ, quæ ex punctis intra duo polygona ſimilia & æqualia ſimiliter locatis e*:untes angulos adeunt, *maximum* efficient aggre-

gregatum præ rectis analogis ejusdem numeri, quæ consimili positione in duobus aliis pariter polygonis similibus duo dicta polygona simul æquantibus, donantur.

3. Ob $EF . \overline{GI}^{n} < FD . \overline{AB}^{n}$ (*per lemmatis coroll.*), erit quoque $2CF . \overline{KL}^{n} < CD . AB^{n} + CE . \overline{GI}^{n}$, seu statuendo $CD = AB$, erit generatim $2\overline{KL}^{n+1} < \overline{AB}^{n+1} + \overline{GI}^{n+1}$, si $\overline{AB}^{n}$, $\overline{KL}^{n}$, $\overline{GI}^{n}$ in proportione arithmetica progrediantur.

4. Quia $2\overline{KL}^{n+1} < \overline{AB}^{n+1} + \overline{GI}^{n+1}$, si $\overline{AB}^{n}$, $\overline{KL}^{n}$, $\overline{GI}^{n}$ sint in proportione arithmetica; e contrario erit $2\overline{KL}^{n} > \overline{AB}^{n} + \overline{GI}^{n}$, si $\overline{AB}^{n+1}$, $\overline{KL}^{n+1}$, $\overline{GI}^{n+1}$ in proportione arithmetica incedant. Sic ponendo $n = 2$, prodit $2\overline{KL}^{3} < \overline{AB}^{3} + \overline{GI}^{3}$, si $\overline{AB}^{2}$, $\overline{KL}^{2}$, $\overline{GI}^{2}$ progressionem arithmeticam servent; ergo ut fiat $2\overline{KL}^{3} = \overline{AB}^{3} + \overline{GI}^{3}$, oportet augere duos cubos similes, & æquales $\overline{KL}^{3}$, $\overline{KL}^{3}$; tunc autem $2\overline{KL}^{2}$ non est amplius $= \overline{AB}^{2} + \overline{GI}^{2}$, sed major, ut patet.

5. Esto triangulum rectangulum isosceles ABC (*Fig.* 49.) & parabola primi ordinis AMO concava versus axem AB illud secet in puncto N. Agatur semiordinata QN, cui ad æqualia intervalla PQ, QR ducantur aliæ semiordinatæ parallelæ PM, RT tam latus AC in punctis S, N, T, quam parabolam in M, N, O abscindentes. Quoniam AP, AQ, AR ponuntur in progressione arithmetica, in eadem proportione erunt PS, QN, RT, nec non $\overline{PM}^2$, $\overline{QN}^2$, $\overline{RO}^2$. Habemus igitur $2QN = PS + RT$, & $2\overline{QN}^2 = \overline{PM}^2 + \overline{RO}^2$; adeoque, vi corollarii præcedentis, $2QN > PM + RO$.

Sint præterea parabola primi ordinis ASO (*Fig.* 50.) & parabola secundi ordinis AMT sub eadem parametro, in quibus ordinatim applicatarum potestates sint abscissis proportionales; pronum est, eas invicem intersecari in puncto N, ubi parameter alterutras coordinatas AQ, QN adæquat, quod ob eandem rationem in omnibus accidit parabolis ordinum superiorum, applicatarum potestates abscissis proportionales habentibus. Exacta ut supra constructione, ob $2\overline{QN}^3 = \overline{PM}^3 + \overline{RT}^3$, & $2\overline{QN}^2 = \overline{PS}^2 + \overline{RO}^2$, erit per coroll. antec. $2\overline{QN}^2 > \overline{PM}^2 + \overline{RT}^2$. Eodem modo si parabola AMT sit uno ordine superior parabola ASO, sitque $2\overline{QN}^{n+1} =$

$= \overline{PM}^{n+1} + \overline{RT}^{n+1}$, erit $2\overline{QN}^{n} > \overline{PM}^{n} + \overline{RT}^{n}$, idque in infinitum. Quapropter data parabola *AMT* cujuscumque ordinis, in qua ordinatim applicatarum potestates integræ sint abscissis respondentibus proportonales, hoc est $\overline{PM}^{n+1} = AP$; dupla potestas *n* applicatæ *QN* parametro æqualis, nimirum $2\overline{QN}^{n}$ erit præ aggregato similium potestatum ex duabus applicatis *PM*, *RT* per æqualia intervalla *PQ*, *QR* ab ipsa *QN* distantibus, *maximum.*

Scholion I. Idem prosequor argumentum; methodum ab assueta diversam, eamque simplicem pro Tironibus exponens.

Inventa functione unius variabilis ad *maximum* vel *minimum* attinente, pro hac variabili, puta x, substitue $x \pm dx$, & quoniam sumi potest $x = x \pm dx$, æquationem instrue, tum communibus deletis, & dx^2, dx^3, &c. pro nihilo habitis (nam dx^2; dx^3; &c. cum dx comparati evanescunt) quantitas superstes remanebit $= 0$, quæ reducta *maximi*, vel *minimi* valorem suppeditabit. Ratio patet; etenim ob $\infty : 1 = 1 : 0$, si quantitas finita infinito addita vel ab eo detracta nihil immutat, ut in *Introduct.* demonstravi, quantitas quoque infinitesima (quam ostendi æquivalere nihilo) finito addita, vel ab ipso dempta nihil variat. Præterea quia *maximum*, vel *minimum* consideratur velut maxima, vel minima curvæ alicujus ordinata; quum alia ordinata eidem infinite proxima per quantitatem infini-

 nite

nitesimam differat, ambæ ut æquales tuto accipi possunt; quare si maxima ordinata dicatur x, infinite proxima erit $= x - dx$; & contra si ita vocetur x, erit maxima ordinata $= x + dx$, ac proinde fiet $x = x - dx$, vel $x = x + dx$.

Exemplum I. *Rectam* AC (Fig. 21.) *ita secare in* B, *ut* $\overline{AB}^m + n\overline{BC}^m$ *sit* minimum.

Resolutio. Fiat $AC = a$, $AB = x$, erit $BC = a - x$; quare $x^m + n(a-x)^m$ debet esse *minmum.* In locum x suffice $x \pm dx$, & æquationem dispone;

stabit $x^m + n(a-x)^m = (x \pm dx)^m + n(a-x \mp dx)^m$, nimirum peracta evolutione, factisque dx^2, dx^3, &c. nihilo æqualibus, $x^m + n(a-x)^m = x^m \pm mx^{m-1} dx + n(a-x)^m \mp mn(a-x)^{m-1} dx$, ablatisque communibus, restat $o = \pm mx^{m-1}dx \mp mn(a-x)^{m-1}dx$, seu per mdx dividendo, $x^{m-1} = n(a-x)^{m-1}$; extrahe radicem $m-1$, ac permuta, tandemque habebis $x = \dfrac{n^{\frac{1}{m-1}}}{1+n^{\frac{1}{m-1}}} a$. *Quod erat* &c.

Nonnulla interim animadverte. (1.°) Si ordinatam *maximum*, vel *minimum* repræsentantem statuas $= y$, erit in hoc Exemplo $dy = mx^{m-1} dx - mn(a-x)$

$(a-x)^{m-1}dx = 0$, adeoque $\frac{dy}{dx} = mx^{m-1} - mn$ $(a-x)^{m-1} = 0$, ex quo apparet, formulam $\frac{dy}{dx} = 0$, vel $\frac{dx}{dy} = \infty$ *maximis* vel *minimis* inveniendis inservire, ut vulgata Regula docet. (2.°) Per hanc solam methodum non solum quantitatis $x^{\pm m}$, sed ceterarum quoque quantitatum $x^m y^n$, $\frac{x^m}{y^n}$, &c. singula differentialia separatim, facile, ac tutissime obtinentur. Etenim ponendo $x^{\pm m} = (x + dx)^{\pm m}$, & operando ut supra, fit $x^{\pm m} = x^{\pm m} \pm mx^{\pm m-1} dx$, & $0 = \pm mx^{\pm m-1} dx$. Similiter ob $x^m y^n = (x + dx)^m (y + dy)^n = (x^m + mx^{m-1} dx)(y^n + ny^{n-1} dy) = x^m y^n + nx^m y^{n-1} dy + my^n x^{m-1} dx$, habetur $0 = nx^m y^{n-1} dy + my^n x^{m-1} dx$; quare si $m = 1 = n$, erit $d.\,xy = xdy + ydx$. Tandem ob $\frac{x^m}{y^n} =$

$= \frac{(x + dx)^m}{(y + dy)^n} = \frac{x^m + mx^{m-1} dx}{y^n + ny^{n-1} dy}$, reducendo ad idem nomen, permutando, & dividendo per y^{2n}, prodit $\frac{x^m}{y^n} = \frac{x^m}{y^n} +$ $\frac{my^n x^{m-1} dx - nx^m y^{n-1} dy}{y^{2n}}$; ex quo $0 =$ $\frac{my^n x^{m-1} dx - nx^m y^{n-1} dy}{y^{2n}}$ adeoque si $m = 1$ $= n$; stabit $d.\frac{x}{y} = \frac{ydx - xdy}{yy}$. (3.°) Si duæ sint variabiles, puta x, y, reperto per idoneam æquationem ipsius dy valore in terminis specie dx affectis, hunc substitue, & prout dictum est reliqua perage. (4.°) Si Tironi calculo differentiali nondum assueto species dx negotium facessat, is pro hac aliam quamlibet sufficiat, velut ϵ, ac terminis, ubi adsunt ϵ^2, ϵ^3, &c. deletis, vel brevius non computatis, tum reliquis operationibus ad finem perductis, voti compos fiet, ut docet sequens

Exemplum II. *Solidum datum in duo solida similia dispertiri, in quibus superficierum aggregatum sit* maximum.

Resolutio. Alter ex hujusmodi solidis ponatur $= x^3$, alter $= y^3$, tum fiat $x = x + \epsilon$, & $y = y + \omega$. Erit $x^3 + y^3 = x^3 + 3x^2\epsilon + y^3 + 3y^2\omega$; hinc $0 =$

$= 3x^2\varepsilon + 3y^2\omega$, & $\omega = -\frac{x^2\varepsilon}{yy}$; sed $x^2 + y^2$ debet esse *maximum*; ergo $x^2 + y^2 = x^2 + 2x\varepsilon + y^2 + 2y\omega$; ex quo $o = 2x\varepsilon + 2y\omega$, & $\omega = -\frac{x\varepsilon}{y}$. Habemus igitur $\frac{x^2\varepsilon}{yy} = \frac{x\varepsilon}{y}$, ac tandem $x = y$. Ergo solidum, de quo est quæstio, in duo solida similia & æqualia est dividendum. *Quod erat* &c.

Atque ista Tironi in præsentiarum sufficiant.

Scholion II. Meam proposui (*Introd.* §. *639*. n. *6*.) opinionem circa quæstionem, an dentur quantitatum negativarum logarithmi; pauca nunc mihi liceat adiicere. Demonstravi (l. cit. §. *642*.) post cel. Joh. Bernoullium, esse circumferentiam ad diametrum, ut $l - 1$ ad $\sqrt{-1}$; at si $l - 1$ ponatur quantitas realis, circumferentia foret ad diametrum, ut quantitas realis ad imaginariam; quod est absurdum. Si $l - 1$ vices hic gerat quantitatis imaginariæ, duas eadem quantitas induet personas se contradicentes; alterum absurdum. Ergo vel dictorum logarithmorum realitas, vel Bernoulliana demonstratio est reiicienda.

FINIS.

ER-

		ERRATA	CORRIGE.
Pag.	V.		
9	13	$\frac{c}{a}^{n-2}$	$\frac{c}{a} x^{n-2}$.
16	4	$-\frac{cx^2}{d}$	$+\frac{cx^2}{d}$.
15	21	ad ſecundam	ad ſecundum.
45	7	demptor	dempto.
47	9	$x^3 - m^2$	$x^3 - mx^2$.
83	29	concludere	excludere.
140	3	multiplica	multiplico.
157	24	numerus	numeros.
159	7	eliiciemus	eliciemus.
192	21	x^7	x^{27}.
167	22	ſuperſt	ſupereſt.
201	18	libet	liber.
208	22	*AH*	*PB*.
Ibid.	23	*AP.AH*	*AP.PB*.
222	27	$y + \frac{p}{}$	$y + \frac{p}{y}$.
223	4	æquationes	æquationis.
225	18	rectangolo	rectangulo.
231	12	eadem	eædem.
250	15	unum	unus.
274, 275, 288 in locum *v* pone V.			
329	11	denominatore	denominatorem.

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337	1	Hac	Hæ.
341	4	ponatur	ponantur,
350	23	impressæ	expressæ.
435	3	fit vera	sit vera,
441	7	alteratim	alternatim,
445	15	proposita	propositæ,
464	13	propriæ	proprie.
482	15	prædictæ	præditæ.
495	23	æquatione	æquationes.
513	12	$y\frac{2}{5} = x$; rursus	$y\frac{2}{5} = x$, rursus.
525	10	radicum	radium.
540	2	arculus	circulus.
543	21	*Pn*, *Pn*	*PN*, *Pn*,
556	2	experietur	exploretur.

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